\[ J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(1+\nu+j)} \left(\frac{x}{2}\right)^{2j} \]

\[ x \rightarrow 0 \quad \frac{1}{\Gamma(1+\nu)} \left(\frac{x}{2}\right)^\nu \]

\[ x \gg 1 \quad \sqrt{\frac{2}{\pi x}} \cos \left(x - \frac{\nu}{2} \right) \frac{\pi}{4} \]

\[ = \frac{1}{2\pi x} \int_0^{2\pi} d\phi \exp(ix\cos\phi - iv\phi) \quad \text{integer } \nu \]

\[ N_\nu(x) = \frac{1}{\sin \nu \pi} (J_{\nu}(x) \cos \nu \pi - J_{-\nu}(x)) \quad \text{a.k.a. } Y_\nu(x) \]

\[ x \rightarrow 0 \quad -\frac{\Gamma(\nu)}{\pi} \left(\frac{2}{x}\right)^\nu \quad \text{or} \quad \frac{2}{\pi} \left\{ \ln \left(\frac{x}{2}\right) + \gamma_{\text{Euler}} \right\} \]

\[ x \gg 1 \quad \sqrt{\frac{2}{\pi x}} \sin \left(x - \frac{\nu}{2} \right) \frac{\pi}{4} \]

\[ \frac{2}{\pi x} = W \{ J_\nu(x), N_\nu(x) \} \]

where

\[ \ln \Gamma(1+x) = -\ln(1+x) + (1 - \gamma_{\text{Euler}}) x + \sum_{n=2}^{\infty} (-1)^n [\zeta(n) - 1] x^n / n \]

for \( |x| < 2 \) with \( \gamma_{\text{Euler}} = 0.57721 \ 56649 \ \ldots \). For the modified functions,

\[ I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(1+\nu+j)} \left(\frac{x}{2}\right)^{2j} = i^{-\nu} J_\nu(ix) \]

\[ x \rightarrow 0 \quad \frac{1}{\Gamma(1+\nu)} \left(\frac{x}{2}\right)^\nu \]

\[ x \gg 1 \quad \sqrt{\frac{1}{2\pi x}} \exp(x) \]

\[ K_\nu(x) = \frac{\pi}{2\sin \nu \pi} (I_{-\nu}(x) - I_\nu(x)) \]

\[ x \rightarrow 0 \quad \frac{\Gamma(\nu)}{2} \left(\frac{2}{x}\right)^\nu \quad \text{or} \quad -\left\{ \ln \left(\frac{x}{2}\right) + \gamma_{\text{Euler}} \right\} \]

\[ x \gg 1 \quad \sqrt{\frac{\pi}{2x}} \exp(-x) \]

\[ \frac{1}{x} = W \{ K_\nu(x), I_\nu(x) \} \]
Bessel’s differential equation is

\[ x \frac{d}{dx} \left( x \frac{d}{dx} I_\nu \right) = \left( \nu^2 - x^2 \right) I_\nu \]

The modified differential equation is obtained by replacing \( x \rightarrow ix \).

\[ x \frac{d}{dx} \left( x \frac{d}{dx} I_\nu \right) = \left( \nu^2 + x^2 \right) I_\nu \]

Alternatively, upon letting \( s = \ln x \)

\[ \frac{d^2}{ds^2} I_\nu = \left( \nu^2 + e^{2s} \right) I_\nu \]

The asymptotic behavior of the solution is straightforward for this case, since we can easily discern dominant terms. Clearly, for large \( s \), assuming that \( I \) and its derivatives are also large, we find \( I_\nu \sim e^{s} = e^x \). Writing \( I_\nu = f(s) e^{s} \) then transforms the differential equation into \( f'' - \nu^2 f + e^s (2f' + f) = 0 \), which for large \( s \) gives \( 2f' \sim -f \), and hence \( \frac{d}{ds} \ln f \sim -1/2 \). Thus \( f \sim \exp(-s/2) = 1/\sqrt{x} \), and so \( I_\nu \sim e^x/\sqrt{x} \). Now, there is no way to obtain the normalization factor \( 1/\sqrt{2\pi} \) just from the asymptotic behavior of the differential equation, since the equation is linear in \( f \). It is necessary to connect the large \( x \) behavior to the small \( x \) behavior, where we have fixed the normalization through that of the above power series. We will be content to check the normalization numerically, for \( I_0 \), as given below.
$J_0(x)$ and $Y_0(x)$ (or $N_0(x)$)

$J_0(x)$ and $\sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi}{2} \right)$
$J_1(x)$ and $Y_1(x)$ (or $N_1(x)$)

$J_1(x)$ and $\sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi}{2} - \frac{\pi}{4} \right)$
$I_0(x)$ and $K_0(x)$

$I_1(x)$ and $K_1(x)$