Angular method to calculate the electric field of a long cylindrical shell

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Abstract
A new method to calculate the electric field inside a long cylindrical shell with surface charge density $\sigma = \sigma_0 \cos \theta$ is presented. The integral can be readily carried out without invoking special functions typically used for this problem. In the case of constant surface charge density, a simple geometry can be used to demonstrate the cylindrical shell theorem with elementary calculus.

Introduction
The electric field inside a uniformly charged long cylindrical is a standard example discussed in the introductory university physics. The field is zero inside the cylinder, outside of the cylinder the field behaves like a long wire at the center carrying the same amount of charge per unit length. It is usually proven with the use of Gauss’s law and cylindrical symmetry of the electric field[1-3]. A more direct proof by integration is lacking at this levels. In fact, students often have difficulties to visualize intuitively why the field at off-center points is cancelled out completely and behaves like line charge at the center on the outside.

Another classical electromagnetism problem deals with electric field due to a non-uniform charge density but varying with $\sigma = \sigma_0 \cos \theta$. It describes of charge distribution of several systems such as the bound surface charge density of a uniformly polarized cylinder or the induced surface charge density of a cylindrical conductor inside a uniform electric field. In the case of uniformly polarized sphere $\sigma_0 = p$, $p$ is the dipole moment per unit volume. The calculation of the field is a standard example or problem of boundary value problems, often found in any intermediate electromagnetism books[4-5].

A similar problem exists for the case of spherical geometry where a uniform electric field is produced inside a uniformly polarized spheres. A more familiar one is the field inside and outside of uniformly charged spherical shells.

Here we attempt to demonstrate via an angular method the classical results using elementary calculus only, instead of specialized functions normally required.

Results
Consider a long cylindrical shell with a surface charge density $\sigma = \sigma_0 \cos \theta$ lies along the x-axis. The cross section of the cylinder is in the y-z plane and $\theta$ is defined with respect to the z-axis, as shown in Figure 1. The electric field inside the shell at point P can be considered as superposition of infinitely long lines distributed on the surface of the cylinder in the x-axis direction. It is well known that the electric
field from an infinite long wire of uniform charge density $\lambda$ is $E = \frac{\lambda}{2\pi\varepsilon_0 r}$, here $r$ is the distance from the line. For the geometry considered, the effective charge density for an infinitesimal arc length $dL$ is $\lambda = \frac{dL}{\sigma(\theta)}$, with corresponding electric field contribution $dE = \frac{dL}{2\pi\varepsilon_0 r} = \frac{d\beta}{2\pi\varepsilon_0 \cos \alpha} \sigma(\theta)$ in the $-\hat{r}$ direction ($\hat{r}$ is defined from P to $dL$) $\alpha$ is the angle between $\hat{r}$ and the normal of the arc length. $\beta$ is the angle between $\hat{r}$ and z-axis. $dL$ corresponds to the arc length when the angle changes from $\beta$ to $\beta+d\beta$ and $dL = \frac{rd\beta}{\cos \alpha}$. $dL'$ corresponds to the arc length in the opposite direction. Because of circular geometry $\alpha = \alpha'$. Thus, for uniform surface charge density, the contribution from $dL$ and $dL'$ cancels out completely, resulting a zero electric field inside the long cylinder shell[7].

If the charge density is in the form $\sigma = \sigma_0 \cos \theta$, the net electric field contribution

$$dE_n = dE - dE' = \frac{2kd\beta}{\cos \alpha} [\sigma(\theta) - \sigma(\gamma)]$$

along the $-\hat{r}$ direction with $\gamma = \pi - 2\alpha - \theta$. Because of the symmetry about z-axis, i.e. $\sigma$ is only dependent on the angle $\Theta$, the horizontal component cancels out and leaving only the vertical component:

$$dE_z = -dE_n \cos \beta = -\frac{2kd\beta}{\cos \alpha} \sigma_0 [\cos \theta - \cos \gamma] \cos \beta = -\frac{2kd\beta}{\cos \alpha} \sigma_0 [-2 \sin \frac{\theta - \gamma}{2} \sin \frac{\theta + \gamma}{2}] \cos \beta$$

$$= -\frac{2kd\beta}{\cos \alpha} \sigma_0 2\cos(\theta + \alpha) \cos \alpha \cos \beta = -4k\sigma_0 (\cos \beta)^2 d\beta$$

The total electric is then given by

$$E = E_z = -\int_0^\pi 4k\sigma_0 (\cos \beta)^2 d\beta = -\frac{\sigma_0}{2\varepsilon_0}$$

Thus for a uniformly polarized cylinder with dipole moment per unit volume of $\vec{P}$, the electric field inside the cylinder is $\vec{E} = -\frac{\vec{P}}{2\varepsilon_0}$

This problem is generally discussed in intermediate EM courses and requires special functions to discuss its solutions[4-6].

For uniformly charged long cylindrical shell, it is easy to see the field is zero inside, as discussed above[7]. On the outside of the shell, similar schematics can be used to prove the shell theorem for cylindrical systems.
As shown Figure 2, field at point P can be calculated by integrating the contribution of dL arc along the ring, subtended by an infinitesimal angle dβ. r is the distance between P and M, and r’ is between P and N. α is the angle between OM and PM. \(\frac{rd\beta}{\cos \alpha}\) and \(\frac{r'd\beta}{\cos \alpha}\) are the two infinitesimal arcs that are contributing to the fields at P within \(\beta\) and \(\beta + d\beta\). Because it is infinitely long along the x-axis, the contribution from dL segment is given

\[dE = \frac{dL \sigma}{2\pi \varepsilon_0 r} = \frac{dL' \sigma}{2\pi \varepsilon_0 r'} = \frac{d\beta}{2\pi \varepsilon_0 \cos \alpha}\]

It is clear from this that dL and dL’ contribute exactly the same since they are subtended by the same infinitesimal \(d\beta\). \(\beta\) changes from 0 to \(\beta_o\) with \(\sin \beta_0 = \frac{R}{Z}\), corresponding to the line PT tangent to the ring from point P. The total field is along the z-axis by symmetry, thus

\[E = \int dE_z = 4 \int_0^{\beta_o} \frac{d\beta}{2\pi \varepsilon_0 \cos \alpha} \sigma \cos \beta\]

The factor 4 comes from the four equal contributions in terms of angle \(\beta\). Two from each side of the z-axis and two from the near surface corresponding to \(\theta = 0 \rightarrow \theta = \theta_o = \frac{\pi}{2} - \beta_o\) and the far surface corresponding to \(\theta = \theta_o \rightarrow \theta = \pi\). It should be noted the four quarter-circles do not contribute equally. If we label the distance between M and C as d, OC is perpendicular to the line PMN and \(OC = Z \sin \beta\), then,

\[\cos \alpha = \frac{d}{R} = \sqrt{R^2 - (Z \sin \beta)^2} = \frac{1 - (\frac{Z}{R})^2 \sin \beta^2}.\]

By introducing \(\sin \beta' = \frac{Z}{R} \sin \beta\), \(\sin \beta'_o = \frac{Z}{R} \sin \beta_o = 1\), thus \(\beta'_o = \frac{\pi}{2}\), the electric field can be simplified to

\[E(z) = 4 \int_0^{\beta'_o} \frac{R \, d\beta'}{2\pi \varepsilon_0 \sigma} = \frac{2R \sigma \beta'_o}{2\pi \varepsilon_0} = \frac{\sigma 2R\pi}{2\pi \varepsilon_0 Z} = \frac{\lambda}{2\pi \varepsilon_0 Z}\]

This is the result of a field due to a long wire with line charge density of \(\lambda = \sigma 2R\pi\) at a distance \(Z\) from the wire.

**Conclusion**

In summary, we have presented an easily integrable approach to calculate the electric field inside a long cylindrical shell of charge density \(\sigma = \sigma_o \cos \theta\) and a vigorous proof of shell theorem for a uniformly charged long cylinder. To our knowledge, this is a new method to solve this geometry and the present approach is not known in the literature if published somewhere. The approach is vigorous and yet manageable with elementary calculus without the use of special functions for these classical EM problems[8].

It should also be noted there is indeed alternative method to prove the uniform field inside the uniformly polarized cylinder by considering superposition of two oppositely charged cylinders (q per unit volume) separated by an infinitesimal distance d but with a finite dipole moment of \(P=qd[4-5]\).
References


