Relativistic E&M Conventions

Coordinate component and Lorentz metric conventions.

$$x^\alpha = (ct, \vec{x}) \ , \quad x_\alpha = (ct, -\vec{x}) \ , \quad x^\alpha x_\alpha = c^2 t^2 - \vec{x} \cdot \vec{x} \ , \quad \partial_\alpha = \left(\frac{1}{c}\, \partial_t, \vec{\nabla} \right) \ , \quad \partial^\alpha = \left(\frac{1}{c}\, \partial_t, -\vec{\nabla} \right) \ , \quad \Box = \partial^\alpha \partial_\alpha = \frac{1}{c^2}\, \partial_t^2 - \vec{\nabla} \cdot \vec{\nabla}$$

$$g^{\alpha\beta} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ , \quad g_{\alpha\beta} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

I may often write $x \cdot y$ instead of $x^\alpha y_\alpha$. This should be easy to distinguish from the Euclidean dot product of the spatial components $\vec{x} \cdot \vec{y}$ in a given context. Proper time and distance are:

$$c^2 d\tau^2 \equiv ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = c^2 \left(dt \right)^2 - d\vec{x} \cdot d\vec{x}$$

Lorentz transformation between two observors' frames: x and X.

$$\begin{aligned} X^\alpha &= \Lambda^\alpha{}_\beta x^\beta \ , \quad x^\alpha = \left(\Lambda^{-1} \right)^\alpha{}_\beta X^\beta \\ \Lambda^\alpha{}_\beta &= \frac{\partial X^\alpha}{\partial x^\beta} \ , \quad \left(\Lambda^{-1} \right)^\alpha{}_\beta = \frac{\partial x^\alpha}{\partial X^\beta} \\ dX^\alpha &= \Lambda^\alpha{}_\beta dx^\beta \ , \quad dx^\alpha = \left(\Lambda^{-1} \right)^\alpha{}_\beta dX^\beta \\ \frac{\partial}{\partial X^\alpha} &= \left(\Lambda^{-1} \right)^\beta{}_\alpha \frac{\partial}{\partial x^\beta} \ , \quad \frac{\partial}{\partial x^\alpha} = \Lambda^\beta{}_\alpha \frac{\partial}{\partial X^\beta} \end{aligned}$$

First line here is valid only for *linear* transformations. Other lines are true more generally. For example, when the frames differ by a constant velocity $\vec{v} = v\, \hat{z} \equiv \hat{z}\, c \tanh \theta$

$$\begin{pmatrix} cT \\ X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cosh\theta & 0 & 0 & -\sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\theta & 0 & 0 & \cosh\theta \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

where $\cosh\theta = \gamma \equiv 1/\sqrt{1-v^2/c^2}$ and $\sinh\theta = \gamma v/c$. If the X^α frame is comoving with a particle, then the particle's proper time is given by

$$d\tau = \frac{1}{\gamma}\ dt$$

Currents, potentials, and fields. In SI units, in vacuum,

$$\vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A}\ ,\quad \vec{B} = \vec{\nabla}\times\vec{A}\ ,\quad \vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\ \rho\ ,\quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\ \partial_t\vec{E}$$

etc. These become

$$J^\alpha = \left(c\rho, \vec{J}\right)\ ,\quad A^\alpha = \left(\frac{\Phi}{c}, \vec{A}\right)$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}\ ,\quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

where $\vec{E} = E_x\ \widehat{x} + E_y\ \widehat{y} + E_z\ \widehat{z}$, etc. Current conservation $\partial_t\rho + \vec{\nabla}\cdot\vec{J} = 0$ is $\partial_\alpha J^\alpha = 0$, and the Lorenz condition $\frac{1}{c^2}\partial_t\Phi + \vec{\nabla}\cdot\vec{A} = 0$ is $\partial_\alpha A^\alpha = 0$.

Energy and momentum for point particle of mass m.

$$p^\alpha = \left(\frac{E}{c}, \vec{p}\right) = (\gamma mc, \gamma m\vec{v})\ ,\quad p_\alpha = \left(\frac{E}{c}, -\vec{p}\right)\ ,\quad p^\alpha p_\alpha = m^2c^2$$

For charge q and mass m, the Lorentz force equation, $\frac{d}{d\tau}\vec{p} = q\ \vec{E} + \frac{q}{m}\ \vec{p}\times\vec{B}$, and corresponding rate of change of the particle's energy, $\frac{d}{d\tau}E = \frac{q}{m}\ \vec{p}\cdot\vec{E}$, become

$$\frac{d}{d\tau}p^\alpha = \frac{q}{m}\ F^{\alpha\beta}p_\beta$$