

PHY651 Midterm Exam  
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This is a take-home, open textbook exam (Panofsky and Phillips, *only*).

However, if you use results from the text, you must indicate precisely what is being used and where it is in the text (for example, give the formula number). And no credit will be given for quoting a textbook result *if* you are asked in the exam problem to provide the derivation of that result. (Obviously!)

You may also consult any lecture notes taken in class this semester.

You may **not** discuss the exam with anyone except Professor Curtright.

**No credit** will be given **unless** you work in SI units.

Good luck!

**“On my honor, I have neither received nor given aid on this exam.”**

Signature:

Print Name:

Student ID Number:

**Problem 1** *A modicum of understanding*

For monochromatic electric and magnetic fields

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}) e^{-i\omega t}, \quad \vec{B}(\vec{r}, t) = \vec{B}(\vec{r}) e^{-i\omega t} \quad (1)$$

the relations between the fields and potentials are

$$\vec{E}(\vec{r}) = -\nabla\Phi(\vec{r}) + i\omega\vec{A}(\vec{r}), \quad \vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r}) \quad (2)$$

To save some writing here, we have suppressed the frequency label on all fields, so  $\Phi(\vec{r})$  is actually  $\Phi_\omega(\vec{r})$ , etc.

(a) Starting from Maxwell's equations for monochromatic fields *and* current/charge sources, derive the inhomogeneous Helmholtz equations obeyed by potentials  $\vec{A}(\vec{r})$  and  $\Phi(\vec{r})$  when subject to the Lorenz gauge condition:  $\nabla \cdot \vec{A}(\vec{r}) = \frac{i\omega}{c^2}\Phi(\vec{r})$ .

(b) What are the integral solutions for the potentials that behave as  $\sim \frac{1}{r} \exp(ikr)$  when all sources vanish rapidly for large  $r$ ?

(c) From the integral solution for  $\vec{A}(\vec{r})$  obtain the corresponding integral solution for  $\vec{B}(\vec{r})$ .

(d) From your answer to (c), Fourier re-compose the fields and sources to obtain the "retarded" time-dependent integral solution for general  $\vec{B}(\vec{r}, t)$ .

**Problem 2** *The straight and skinny*

Point charges  $\pm q(t)$  are at locations  $z = \pm a$ , with a straight filamentary wire connecting the two charges. The wire carries a current  $i(t) = I + Wt$  in the  $z$ -direction such that the the point charges vary with time according to

$$q(t) = Q + It + \frac{1}{2}Wt^2 \quad (3)$$

with constant  $Q$ ,  $I$ , and  $W$ .

(a) Determine  $\vec{B}(\vec{r}, t)$ .

(b) Are there any “retardation effects” in your final answer?

(c) Also, determine the electromagnetic energy flow in the  $z$ -direction by computing the Poynting component  $S_z(\vec{r}, t)$ , for large  $r$ .

Note: The following integrals might be useful.

$$\int \frac{1}{(r^2 + s^2 - 2zs)^{1/2}} ds = \ln \left( s - z + \sqrt{r^2 + s^2 - 2zs} \right)$$
$$\int \frac{1}{(r^2 + s^2 - 2zs)^{3/2}} ds = \frac{s - z}{(r^2 - z^2) \sqrt{r^2 + s^2 - 2zs}}$$

**Problem 3** *A light vector workout*

A multipole field is known to be purely  $l, m = 1, 0$  with magnetic field given in a sourceless region by

$$\vec{H}(\vec{r}) = H j_1(kr) \vec{Y}_{1,0}(\hat{r}) \quad (4)$$

where  $H$  is a constant. Recall that  $j_1(s) = \frac{\sin s}{s^2} - \frac{\cos s}{s}$  and  $Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta$ .

- (a) Is this an electric multipole (TM), or a magnetic multipole (TE) field? Why?
- (b) What are the various components of  $\vec{H}(\vec{r})$ ?
- (c) What are the components of the corresponding electric field in the same sourceless region?

**Problem 4** *E and B nicely packaged together*

The covariant wave equation satisfied by the field strength, in the presence of a given 4-current  $J_\mu(x)$ , is

$$\partial^\lambda \partial_\lambda F_{\mu\nu}(x) = \mu_0 (\partial_\mu J_\nu(x) - \partial_\nu J_\mu(x)) \quad (5)$$

(a) Assuming the given current and its partial derivatives are highly localized in both space and time, construct the “time-delayed” solution of this equation as an integral involving the current and the covariant form of the retarded Green’s function. You may assume that there are no “incoming” free fields unrelated to  $J_\mu$ .

(b) For a point particle of mass  $m$  and charge  $q$  with a given trajectory,  $X^\mu(\tau)$ , parameterized by the particle’s proper time  $\tau$ , show that your solution of part (a) becomes

$$F_{\mu\nu}(x) = \frac{q}{4\pi\epsilon_0} \frac{1}{(x-X)_\alpha V^\alpha} \frac{d}{d\tau} \left( \frac{(x-X)_\mu V_\nu - (x-X)_\nu V_\mu}{(x-X)_\beta V^\beta} \right) \Bigg|_{\tau=\tau_0} \quad (6)$$

where  $V^\alpha(\tau) = dX^\alpha(\tau)/d\tau$  is the particle’s 4-velocity, and where all terms in the expression are evaluated (*after* differentiating) at the retarded proper time  $\tau_0$  determined as the delayed solution of the light-cone condition  $0 = (x-X(\tau_0))^\lambda (x-X(\tau_0))_\lambda$ . It may be helpful for this part of the problem to note that the 4-current for the point particle can be conveniently written as

$$J_\mu(x) = cq \int d\tau V_\mu(\tau) \delta^4(x-X(\tau)) \quad (7)$$

**Problem 5** *Larmorrrrrrr, Laaaaarmorrr*

Larmor's formula for the instantaneous total radiated power (obtained by integrating the flux over all directions) emitted from an accelerating non-relativistic point particle of mass  $m$  and charge  $q$  is

$$\mathcal{P} = \frac{2}{3} \left( \frac{q^2}{4\pi\epsilon_0 m^2 c^3} \right) \left| \frac{d\vec{p}}{dt} \right|^2 \quad (8)$$

where  $\vec{p} = m\vec{v}$  is the particle's non-relativistic momentum.

(a) *Given* that this power is the non-relativistic limit of a Lorentz scalar, obtain the *unique*, manifestly Lorentz invariant form for  $\mathcal{P}$  in terms of the particle's 4-momentum  $p_\mu$  and/or the proper-time derivatives  $dp_\mu/d\tau$ . Consider and explain how to rule out all other invariants constructed from  $p_\mu$  and/or  $dp_\mu/d\tau$ .

(b) Show in complete detail that your Lorentz invariant answer for  $\mathcal{P}$  is exactly the same as "Liénard's formula," valid for relativistic motion:

$$\mathcal{P} = \frac{2}{3} \left( \frac{q^2}{4\pi\epsilon_0 c^3} \right) \left\{ \gamma^4 |\vec{a}|^2 + \gamma^6 (\vec{a} \cdot \vec{v}/c)^2 \right\} \quad (9)$$

where  $\vec{v}$  is the particle's instantaneous velocity,  $\vec{a} = d\vec{v}/dt$ , and  $\gamma = 1/\sqrt{1 - v^2/c^2}$  is the particle's Lorentz factor.

**Problem 6** *A first look at radiation reaction*

For a point particle of invariant mass  $m$ , Newton's equations

$$m \frac{d^2 \vec{x}}{dt^2} = \vec{F} \quad (10)$$

can be generalized to the Lorentz covariant form

$$\frac{dp^\mu}{d\tau} = F^\mu \quad (11)$$

provided the force  $\vec{F}$  has been carefully incorporated into a 4-vector  $F^\mu$ , whose spatial part reduces to  $\vec{F}$  in the non-relativistic limit, and which in general satisfies the constraint

$$p_\mu F^\mu = 0 \quad (12)$$

This constraint is required by the invariant mass condition,  $p_\mu p^\mu = m^2 c^2$ , from which it follows that  $p_\mu dp^\mu / d\tau = 0$ .

With the same provisos, the "Abraham-Lorentz equation" which includes a force due to "radiation reaction" is

$$m \frac{d^2 \vec{x}}{dt^2} = m\tau \frac{d^3 \vec{x}}{dt^3}, \quad \tau \equiv \frac{2}{3mc^3} \frac{q^2}{4\pi\epsilon_0} \quad (13)$$

This can be generalized to Lorentz covariant form.

(a) Do so, and verify that your result is consistent with  $p_\mu dp^\mu / d\tau = 0$ .

(b) Specialize to motion in only one spatial dimension, say the  $z$  direction. Solve for  $p_z(\tau)$  in terms of  $p_z(0)$  and  $\frac{dp_z}{d\tau}(0)$ .

**Problem 7** *Special credit*

Consider a simple shift of coordinates

$$x^\mu \rightarrow x^\mu + s^\mu \tag{14}$$

This can be written in the form

$$e^{s^\nu P_\nu} x^\mu = x^\mu + s^\mu \tag{15}$$

where  $P_\nu$  is a first-order differential operator. What is this differential operator?

To make things a little more interesting, consider a “special conformal transformation” of coordinates,

$$x^\mu \rightarrow \frac{x^\mu + x^2 s^\mu}{1 + 2x \cdot s + x^2 s^2} \tag{16}$$

where  $x^2 \equiv x^\nu x_\nu$ ,  $s^2 \equiv s^\nu s_\nu$ , and  $x \cdot s = x^\nu s_\nu$ . This also can be written as

$$e^{s^\nu K_\nu} x^\mu = \frac{x^\mu + x^2 s^\mu}{1 + 2x \cdot s + x^2 s^2} \tag{17}$$

where  $K_\nu$  is an *explicitly*  $x$ -dependent, first-order differential operator.

(a) What is  $K_\nu$  here? Justify your answer.

(b) Explicitly check your answer by verifying that when  $e^{s^\nu K_\nu}$  acts on  $x^\mu$  it gives  $\frac{x^\mu + x^2 s^\mu}{1 + 2x \cdot s + x^2 s^2}$  up to and including all terms of second order in  $s^\alpha$ .

(c) Prove that your result for  $K_\nu$ , when exponentiated as in (17), gives the right-hand-side of (17) to all orders in  $s^\alpha$ .