PHY651 Final Exam
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This is an open textbook exam (only Jackson, 3rd Edition).

If you use results from the text, you must indicate precisely what is being used and where it is in the text (for example, give the formula number). You cannot just quote a result from Jackson and expect any credit if you are asked in the exam problem to provide the derivation of that result. (Obviously!)

You may also consult any lecture notes taken in class this semester.

You may **not** discuss the exam with anyone except Professor Curtright.

**No credit** will be given for problems 2-5 **unless** you work in SI units.

Good luck!

**"On my honor, I have neither received nor given aid on this exam."**

Signature:

Print Name:

Student ID Number:

**Problem 1**

Under a special conformal transformation of coordinates,

$$x^\mu \to \frac{x^\mu + x^2 s^\mu}{1 + 2x \cdot s + x^2 s^2} \tag{1}$$

where $x^2 \equiv x^\nu x_\nu$ , $s^2 \equiv s^\nu s_\nu$ , and $x \cdot s = x^\nu s_\nu$ . This can be written as

$$e^{s^\nu K_\nu}\ x^\mu = \frac{x^\mu + x^2 s^\mu}{1 + 2x \cdot s + x^2 s^2} \tag{2}$$

where $K_\nu$ is an explicitly $x$-dependent, first-order differential operator.

(a) What is $K_\nu$ here? Justify your answer.

(b) Explicitly check your answer by verifying that when $e^{s^\nu K_\nu}$ acts on $x^\mu$ it gives $\dfrac{x^\mu + x^2 s^\mu}{1 + 2x \cdot s + x^2 s^2}$ up to and including all terms of second order in $s^\alpha$.

(c) Prove that your result for $K_\nu$, when exponentiated as in (2), gives the right-hand-side of (2) to all orders in $s^\alpha$.

**Problem 2** (Make sure you use SI units throughout this problem.)

The covariant wave equation satisfied by the field strength, in the presence of a given 4-current $J_\mu(x)$, is

$$\partial^\lambda \partial_\lambda F_{\mu\nu}(x) = \mu_0 \left(\partial_\mu J_\nu(x) - \partial_\nu J_\mu(x)\right) \tag{3}$$

(a) Assuming the given current and its partial derivatives are highly localized in both space and time, construct the "time-delayed" solution of this equation as an integral involving the current and the covariant form of the retarded Green's function. You may assume that there are no "incoming" free fields unrelated to $J_\mu$.

(b) For a point particle of mass $m$ and charge $q$ with a given trajectory, $X^\mu(\tau)$, parameterized by the particle's proper time $\tau$, show that your solution of part (a) becomes

$$F_{\mu\nu}(x) = \frac{q}{4\pi\varepsilon_0} \frac{1}{(x-X)_\alpha V^\alpha} \frac{d}{d\tau} \left( \frac{(x-X)_\mu V_\nu - (x-X)_\nu V_\mu}{(x-X)_\beta V^\beta} \right)\Bigg|_{\tau=\tau_0} \tag{4}$$

where $V^\alpha(\tau) = dX^\alpha(\tau)/d\tau$ is the particle's 4-velocity, and where all terms in the expression are evaluated (*after* differentiating) at the retarded proper time $\tau_0$ determined as the delayed solution of the light-cone condition $0 = (x - X(\tau_0))^\lambda (x - X(\tau_0))_\lambda$. It may be helpful for this part of the problem to note that the 4-current for the point particle can be conveniently written as

$$J_\mu(x) = cq \int d\tau \; V_\mu(\tau) \; \delta^4(x - X(\tau)) \tag{5}$$

(c) By considering the components of $F_{\mu\nu}$ (i.e. $\vec{E}$ and $\vec{B}$) and of $J_\mu$ (i.e. $\rho$ and $\vec{J}$), show that your answer to part (a) is the same as Jefimenko's results, as given by (6.55) and (6.56) in Jackson.

**Problem 3** (Make sure you use SI units throughout this problem.)

Larmor's formula for the instantaneous total radiated power (obtained by integrating the flux over all directions) emitted from an accelerating non-relativistic point particle of mass $m$ and charge $q$ is

$$\mathcal{P} = \frac{2}{3}\left(\frac{q^2}{4\pi\varepsilon_0 m^2 c^3}\right)\left|\frac{d\vec{p}}{dt}\right|^2 \tag{6}$$

where $\vec{p} = m\vec{v}$ is the particle's non-relativistic momentum.

(a) *Given* that this power is the non-relativistic limit of a Lorentz scalar, obtain the *unique*, manifestly Lorentz invariant form for $\mathcal{P}$ in terms of the particle's 4-momentum $p_\mu$ and/or the proper-time derivatives $dp_\mu/d\tau$. Consider and explain how to rule out all other invariants constructed from $p_\mu$ and/or $dp_\mu/d\tau$.

(b) Show in complete detail that your Lorentz invariant answer for $\mathcal{P}$ is exactly the same as Liénard's formula, valid for relativistic motion:

$$\mathcal{P} = \frac{2}{3}\left(\frac{q^2}{4\pi\varepsilon_0 c^3}\right)\left\{\gamma^4\left|\vec{a}\right|^2 + \gamma^6\left(\vec{a}\cdot\vec{v}/c\right)^2\right\} \tag{7}$$

where $\vec{v}$ is the particle's instantaneous velocity, $\vec{a} = d\vec{v}/dt$, and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the particle's Lorentz factor.

**Problem 4** (Make sure you use SI units throughout this problem.)

For a point particle of mass $m$, Newton's equations

$$m\frac{d^2\overrightarrow{x}}{dt^2} = \overrightarrow{F} \tag{8}$$

can be generalized to the Lorentz covariant form

$$\frac{dp^\mu}{d\tau} = F^\mu \tag{9}$$

provided the force $\overrightarrow{F}$ has been carefully incorporated into a 4-vector $F^\mu$, whose spatial part reduces to $\overrightarrow{F}$ in the non-relativistic limit, and which in general satisfies the constraint

$$p_\mu F^\mu = 0 \tag{10}$$

This constraint is required by the invariant mass condition, $p_\mu p^\mu = m^2c^2$, from which it follows that $p_\mu dp^\mu/d\tau = 0$.

With the same provisos, the Abraham-Lorentz equation with both radiation reaction and non-electromagnetic force $\overrightarrow{F}$

$$m\frac{d^2\overrightarrow{x}}{dt^2} = m\tau\frac{d^3\overrightarrow{x}}{dt^3} + \overrightarrow{F}\ , \quad \tau \equiv \frac{2}{3mc^3}\frac{q^2}{4\pi\varepsilon_0} \tag{11}$$

can be generalized to Lorentz covariant form.

(a) Do so, for the case of $\overrightarrow{F} = 0$, and verify that your result is consistent with $p_\mu dp^\mu/d\tau = 0$.

(b) For the case $\overrightarrow{F} \neq 0$, specialize to motion in only one spatial dimension, say the $z$ direction, with a non-electromagnetic force given as some arbitrary function of the proper time, $\widehat{z}\cdot\overrightarrow{F} = f(\tau)$, and write a Lorentz covariant form of (11). Solve for $p_z(\tau)$ in terms of $p_z(0)$ for arbitrary $f(\tau)$.

**Problem 5** (Make sure you use SI units throughout this problem.)

Assume a linear, homogeneous, isotropic, causal connection between $\vec{E}$ and $\vec{D}$, as in Jackson §7.10, especially (7.105). Similarly for $\vec{B}$ and $\vec{H}$, but for simplicity suppose the permeability is that of a vacuum, so just assume $\vec{B} = \mu_0 \vec{H}$. Fourier transform the space and time dependence for all quantities, defining

$$\vec{E}(\vec{x}, t) = \frac{1}{(2\pi)^2} \int d\omega \int d^3k \; \vec{E}\left(\vec{k}, \omega\right) \; e^{i\vec{k}\cdot\vec{x} - i\omega t} \tag{12}$$

etc. (We will distinguish space-time fields from their Fourier transforms by indicating the independent variables on which they depend, but we will use the same symbols for both types of fields.) Then the linear relations between fields become (see (7.103) in Jackson)

$$\vec{D}\left(\vec{k}, \omega\right) = \varepsilon(\omega)\, \vec{E}\left(\vec{k}, \omega\right) \;, \quad \vec{B}\left(\vec{k}, \omega\right) = \mu_0 \vec{H}\left(\vec{k}, \omega\right) \tag{13}$$

(a) Fourier transform the expressions of the fields in terms of the potentials

$$\vec{E}(\vec{x}, t) = -\vec{\nabla}\Phi(\vec{x}, t) - \frac{\partial}{\partial t}\vec{A}(\vec{x}, t) \;, \quad \vec{B}(\vec{x}, t) = \vec{\nabla} \times \vec{A}(\vec{x}, t) \tag{14}$$

to obtain the expressions of $\vec{E}\left(\vec{k}, \omega\right)$ and $\vec{B}\left(\vec{k}, \omega\right)$ in terms of $\Phi\left(\vec{k}, \omega\right)$ and $\vec{A}\left(\vec{k}, \omega\right)$. Fourier transform all of Maxwell's equations and use them to obtain *linear* relations between $\Phi\left(\vec{k}, \omega\right)$, $\vec{A}\left(\vec{k}, \omega\right)$, $\rho\left(\vec{k}, \omega\right)$, and $\vec{J}\left(\vec{k}, \omega\right)$.

(b) Solve these linear relations for $\Phi\left(\vec{k}, \omega\right)$ and $\vec{A}\left(\vec{k}, \omega\right)$ in terms of $\rho\left(\vec{k}, \omega\right)$, and $\vec{J}\left(\vec{k}, \omega\right)$, assuming the *Lorentz condition in the medium*: $\vec{k} \cdot \vec{A}\left(\vec{k}, \omega\right) = \mu_0 \, \omega \, \varepsilon(\omega) \; \Phi\left(\vec{k}, \omega\right)$.

Next, consider a point particle of charge $q$ moving through the medium at *constant* velocity $\vec{v}$, so

$$\rho(\vec{x}, t) = q \; \delta(\vec{x} - \vec{v} \, t) \;, \quad \vec{J}(\vec{x}, t) = \vec{v} \rho(\vec{x}, t) \tag{15}$$

(c) Fourier transform these particular charge and current densities to obtain the corresponding $\rho\left(\vec{k}, \omega\right)$ and $\vec{J}\left(\vec{k}, \omega\right)$. Use your results from (b) to find the corresponding $\Phi\left(\vec{k}, \omega\right)$ and $\vec{A}\left(\vec{k}, \omega\right)$.

(d) Use your results for $\Phi\left(\vec{k}, \omega\right)$ and $\vec{A}\left(\vec{k}, \omega\right)$ in (c) to obtain Fourier integral representations of $\Phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$. Evaluate these integrals for the special case that $\varepsilon(\omega)$ is a constant $\varepsilon$, in the situation of "super-luminal" particle motion, where $v^2 > \frac{1}{\mu_0 \varepsilon}$. The particle is moving faster through the medium than would freely propagating light waves, in this situation. (Hint: The Fourier integrands actually have singularities in this special situation. Avoid the singularities and evaluate the integrals by *requiring* that the potentials vanish for $\hat{v} \cdot \vec{x} > vt$.)

(e) Your results for $\Phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$ in the special situation in (d) should have singularities along a cone ("shock front") around the straight-line trajectory of the particle. This is known as the Cherenkov radiation cone. Find the opening angle of this cone from your expressions for $\Phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$.