Vector potentials and magnetic induction in various circumstances (Notes by T Curtright, 10 November 2003)

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

for static, localized current distributions. For conserved currents, $\vec{\nabla} \cdot \vec{J} = 0$, this vector potential satisfies the Coulomb gauge condition $\vec{\nabla} \cdot \vec{A}(\vec{r}) = 0$, upon integration by parts. When the total current is I restricted to a filamentary wire this becomes

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_{\text{wire}} \frac{d\vec{s}}{|\vec{r} - \vec{r}'|}$$

Current ring Consider a filamentary circular current ring of radius a centered on the origin in the xy plane, carrying current I flowing counterclockwise as viewed from $z > 0$. The vector potential is

$$\overrightarrow{A}(\overrightarrow{r}) = A(r, \theta)\ \widehat{e}_\phi$$

$$\begin{aligned} A(r,\theta) &= \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos\phi\ d\phi}{\sqrt{r^2 + a^2 - 2ra\sin\theta\cos\phi}} \\ &= \frac{\mu_0 I a}{4\pi\sqrt{r^2 + a^2}} \int_0^{2\pi} \frac{\cos\phi\ d\phi}{\sqrt{1 - \lambda\cos\phi}} \end{aligned}$$

where

$$\lambda = 2ra\sin\theta / \left(r^2 + a^2\right) \geq 0$$

Note that

$$1 - \lambda = \frac{r^2 - 2ra\sin\theta + a^2}{r^2 + a^2} = \frac{r^2 - 2ra\cos(\theta - \pi/2) + a^2}{r^2 + a^2} = \frac{1}{r^2 + a^2}\left(\overrightarrow{r} - a\frac{(\overrightarrow{r} - z\widehat{e}_z)}{|\overrightarrow{r} - z\widehat{e}_z|}\right)^2 \geq 0$$

The numerator here is just the square of the distance from $\overrightarrow{r}$ to the *closest point* on the circular current loop.

Now Maple says the integral is just

$$\int_0^{2\pi} \frac{\cos\phi d\phi}{\sqrt{1 - \lambda\cos\phi}} = \frac{4}{\sqrt{\lambda(1 - \lambda)}}\left(\text{EllipticK}\left(\sqrt{\frac{2\lambda}{\lambda - 1}}\right) + (\lambda - 1)\,\text{EllipticE}\left(\sqrt{\frac{2\lambda}{\lambda - 1}}\right)\right)$$

where the complete elliptic integrals are defined by

$$\begin{aligned} \int_0^{\pi/2} \sqrt{1 - \kappa\sin^2\phi}\,d\phi &= \text{EllipticE}\left(\sqrt{\kappa}\right) \\ \int_0^{\pi/2} \frac{1}{\sqrt{1 - \kappa\sin^2\phi}}\,d\phi &= \text{EllipticK}\left(\sqrt{\kappa}\right) \\ &= \text{EllipticF}\left(1, \sqrt{\kappa}\right) \end{aligned}$$

For completeness, the incomplete elliptic integrals are given, for $0 \leq \varphi \leq \pi/2$, by

$$\begin{aligned} \int_0^{\varphi} \sqrt{1 - \kappa\sin^2\phi}\,d\phi &= \text{EllipticE}\left(\sin\varphi, \sqrt{\kappa}\right) \\ \int_0^{\varphi} \frac{1}{\sqrt{1 - \kappa\sin^2\phi}}\,d\phi &= \text{EllipticF}\left(\sin\varphi, \sqrt{\kappa}\right) \end{aligned}$$

Now, as written above for the ring situation at hand, our results have purely imaginary arguments for the complete elliptic integrals. However, this is not really indicative of complex results since the elliptic integrals have even series expansions in $\sqrt{\kappa}$:

$$\begin{aligned} \text{EllipticE}\left(\sqrt{\kappa}\right) &= \frac{1}{2}\pi - \frac{1}{8}\pi\kappa - \frac{3}{128}\pi\kappa^2 - \frac{5}{512}\pi\kappa^3 - \frac{175}{32\,768}\pi\kappa^4 - \frac{441}{131\,072}\pi\kappa^5 + O\left(\kappa^6\right) \\ \text{EllipticK}\left(\sqrt{\kappa}\right) &= \frac{1}{2}\pi + \frac{1}{8}\kappa\pi + \frac{9}{128}\kappa^2\pi + \frac{25}{512}\kappa^3\pi + \frac{1225}{32\,768}\kappa^4\pi + \frac{3969}{131\,072}\kappa^5\pi + O\left(\kappa^6\right) \end{aligned}$$

etc. Anyway, a power series is most easily obtained by first expanding the integrand and then performing the integration.

$$\begin{aligned}\frac{\cos\phi}{\sqrt{1-\lambda\cos\phi}} &= \cos\phi+\left(\frac{1}{2}\cos^2\phi\right)\lambda+\left(\frac{3}{8}\cos^3\phi\right)\lambda^2+\left(\frac{5}{16}\cos^4\phi\right)\lambda^3+\left(\frac{35}{128}\cos^5\phi\right)\lambda^4\\ &\quad+\left(\frac{63}{256}\cos^6\phi\right)\lambda^5+\left(\frac{231}{1024}\cos^7\phi\right)\lambda^6+\left(\frac{429}{2048}\cos^8\phi\right)\lambda^7+O\left(\lambda^8\right)\end{aligned}$$

So then

$$\int_0^{2\pi}\frac{\cos\phi\ d\phi}{\sqrt{1-\lambda\cos\phi}}=\frac{1}{2}\lambda\pi+\frac{15}{64}\lambda^3\pi+\frac{315}{2048}\lambda^5\pi+O\left(\lambda^7\right)$$

a series in only odd powers of λ. This comports with letting $\phi\to\phi+\pi$ to obtain

$$\int_{\substack{\text{full}\\ \text{circle}}}\frac{\cos\phi\ d\phi}{\sqrt{1-\lambda\cos\phi}}=-\int_{\substack{\text{full}\\ \text{circle}}}\frac{\cos\phi\ d\phi}{\sqrt{1+\lambda\cos\phi}}$$

at least for real λ, with $-1<\lambda<1$.

Therefore, various forms of the answer for the vector potential are: $\kappa\equiv\sqrt{\frac{4ar\sin\theta}{r^2+a^2+2ar\sin\theta}}$

$$\begin{aligned}A(r,\theta) &= \frac{\mu_0 I}{2\pi r\sin\theta\sqrt{r^2+a^2+2ar\sin\theta}}\left[\left(a^2+r^2\right)\text{EllipticK}(\kappa)-\left(r^2+a^2+2ar\sin\theta\right)\text{EllipticE}(k)\right]\\ &= 2\pi\mu_0 Ia\sum_{(\text{odd})\ l=1}^{\infty}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{l\,1}(\theta,0)\,Y_{l\,1}^*\left(\frac{\pi}{2},0\right)=\frac{\mu_0 Ia}{4}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n-1)!!}{2^n(n+1)!}\frac{r_<^{2n+1}}{r_>^{2n+2}}P_{2n+1}^1(\cos\theta)\\ &= \frac{\mu_0 Ia}{\pi}\int_0^\infty\cos(kz)\,I_1(k\rho_<)\,K_1(k\rho_>)\,dk\\ &= \frac{\mu_0 Ia}{2}\int_0^\infty e^{-k|z|}J_1(ka)\,J_1(k\rho)\,dk\end{aligned}$$

where $r_{\gtrless}=\begin{matrix}\max\\ \min\end{matrix}(a,r)$, $\rho_{\gtrless}=\begin{matrix}\max\\ \min\end{matrix}(a,\rho)$, and where the infinite series/integral forms follow from the various expansions of the elementary Green function.

$$\begin{aligned}\frac{1}{\left|\vec{r}-\vec{r}'\right|} &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}(\theta,\phi)\\ &= \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty e^{im\left(\phi-\phi'\right)}\cos k\left(z-z'\right)I_m(k\rho_<)\,K_m(k\rho_>)\,dk\\ &= \sum_{m=-\infty}^{\infty}\int_0^\infty e^{im\left(\phi-\phi'\right)}e^{-k(z_>-z_<)}J_m(k\rho)\,J_m\left(k\rho'\right)dk\end{aligned}$$

The magnetic induction is now given just by taking the curl of the vector potential.

Straight wire Next, consider a straight segment of wire, carrying current I from $z=-L_1$ to $z=+L_2$. This gives a vector potential

$$A_z(\rho,z)=\frac{\mu_0 I}{4\pi}\left(-2\ln\rho+\ln\left(L_2-z+\sqrt{\rho^2+\left(L_2-z\right)^2}\right)+\ln\left(L_1+z+\sqrt{\rho^2+\left(L_1+z\right)^2}\right)\right)$$

where $\rho=\sqrt{x^2+y^2}$ with the other two components of $\vec{A}$ vanishing. The answer is independent of ϕ, the azimuthal angle about the wire. Note that in the limit of small ρ, or alternatively, in the double limit $L_{1,2}\to\infty$, we have $A_z(\rho,z)=-\frac{\mu_0 I}{2\pi}\ln\rho+f(z)$. The latter "gauge function part" $f(z)$ would not contribute to the magnetic induction.

From this vector potential, we obtain a magnetic induction

$$\begin{aligned}
\vec{B}(\vec{r}) &= \vec{\nabla}\wedge\vec{A}(\vec{r}) = \hat{e}_\phi\left(-\frac{\partial}{\partial\rho}A_z(\rho,z)\right)\\
&= \hat{e}_\phi\frac{\mu_0 I}{4\pi}\left(\frac{2}{\rho}-\frac{\rho/\sqrt{\rho^2+(L_2-z)^2}}{L_2-z+\sqrt{\rho^2+(L_2-z)^2}}-\frac{\rho/\sqrt{\rho^2+(L_1+z)^2}}{L_1+z+\sqrt{\rho^2+(L_1+z)^2}}\right)
\end{aligned}$$

However, this answer is *not* completely physical in the sense that the current for this segment of wire alone is *not* conserved. Instead,

$$\vec{\nabla}\cdot\vec{J}_{\text{wire}} = I\ \delta(x)\,\delta(y)\ \left(\ \delta(z+L_1)-\delta(z-L_2)\ \right)$$

with current diverging from the source point $z=-L_1$, and converging into the sink point $z=+L_2$. As a remedy for this malady, we may suppose the current continues outward from the sink point, say through another wire, or through some ambient conducting material. Similarly, we may suppose the source point has an additional current flowing into it.

For example, ignoring the lower end of the wire, suppose the upper end of the wire terminates on an infinite conducting plane perpendicular to the z axis, with the current flowing into the plane and then uniformly away from the wire as described by

$$\vec{J}(\vec{r}) = I\ \hat{e}_z\ \delta(x)\,\delta(y)\ \theta(L_2-z) + I\ \hat{e}_\rho\ \delta(z-L_2)\ \frac{1}{2\pi\rho}$$

The second term compensates for the divergence of the first. Across such an ideal plane of current, with planar current density $\vec{K}$ amperes/meter, tangential $\vec{B}$ perpendicular to the direction of current flow is discontinuous by an amount $\text{disc}\left(\hat{n}\wedge\vec{B}\right)=\mu_0\vec{K}$, where $\hat{n}$ is a unit vector normal to the plane, as follows from $\vec{\nabla}\wedge\vec{B}(\vec{r})=\mu_0\vec{J}(\vec{r})$. For the plane at the end of the wire under consideration, this implies $\text{disc}(B_\phi) = \lim_{z\to 0+}\left(B_\phi(\rho,L_2-z)-B_\phi(\rho,L_2+z)\right) = \mu_0 I/(2\pi\rho)$, which is satisfied by $B_\phi(\rho,z)=\frac{\mu_0 I}{2\pi\rho}\theta(L_2-z)+$ *const*. Again ignoring the lower end of the wire, we easily check that

$$\mu_0\vec{J}(\vec{r}) = \vec{\nabla}\wedge\vec{B}(\vec{r}) = \hat{e}_\rho\ \left(-\partial_z B_\phi\right)+\hat{e}_z\ \left(\frac{1}{\rho}\partial_\rho\left(\rho B_\phi\right)\right)$$

provided *const* $=0$. This value of *const* is also consistent with the boundary condition $\lim_{\rho\to\infty}B_\phi(\rho,z)=0$.

We conclude the magnetic induction around a semi-infinite $(-\infty<z<L_2)$ wire + plane conductor configuration is discontinuous, equal to just that of an infinite $(-\infty<z<+\infty)$ wire below the plane, and zero above the plane.

$$B_\phi(\rho,z)=\frac{\mu_0 I}{2\pi\rho}\theta(L_2-z)$$

For a vector potential with no ϕ component, and with ρ and z components depending only on ρ and z, we have

$$\vec{B}(\vec{r}) = \vec{\nabla}\wedge\vec{A}(\vec{r}) = \hat{e}_\phi\left(\partial_z A_\rho(\rho,z)-\partial_\rho A_z(\rho,z)\right)$$

The previous B_ϕ follows from taking

$$A_\rho(\rho,z) = -\frac{\mu_0 I}{4\pi}\frac{|z-L_2|}{\rho}\ ,\quad A_z(\rho,z) = -\frac{\mu_0 I}{4\pi}\ln\rho$$

The first of these components is just a vector potential for the ideal plane of current, while the second is *half* the usual vector potential for an infinite $(-\infty<z<+\infty)$ wire, up to irrelevant gauge dependent parts. Our choice for the components is such that the Coulomb gauge condition is satisfied.

$$\vec{\nabla}\cdot\vec{A}(\vec{r}) = \frac{1}{\rho}\partial_\rho\left(\rho A_\rho(\rho,z)\right)+\partial_z A_z(\rho,z) = 0\ .$$

For another very interesting example, suppose this outflow from $z=+L_2$ takes place in a spherically uniform way. Then

$$\vec{J}_{\text{out}}(\vec{r}) = \frac{I}{4\pi}\frac{\vec{r}-L_2\hat{e}_z}{\left|\vec{r}-L_2\hat{e}_z\right|^3} = \frac{-I}{4\pi}\vec{\nabla}\left(\frac{1}{\left|\vec{r}-L_2\hat{e}_z\right|}\right)$$

This compensates for the previous non-zero divergence at the sink point, since

$$\vec{\nabla}\cdot\vec{J}_{\text{out}}\left(\vec{r}\right)=\frac{-I}{4\pi}\nabla^{2}\left(\frac{1}{\left|\vec{r}-L_{2}\hat{e}_{z}\right|}\right)=+I\;\delta\left(\vec{r}-L_{2}\hat{e}_{z}\right)=I\;\delta\left(x\right)\delta\left(y\right)\delta\left(z-L_{2}\right)$$

The total current density is thus conserved at $z=+L_2$, $\vec{\nabla}\cdot\left(\vec{J}_{\text{wire}}+\vec{J}_{\text{out}}\right)=0$. Now, this particular outflow leads to an intriguing problem. What is the vector potential, and what is the magnetic field, for this radial outflow alone?

Strictly speaking, there is none!?!? That is to say, there is no *globally* well-defined non-vanishing $\vec{A}_{\text{out}}\left(\vec{r}\right)$ nor is there a globally defined non-vanishing $\vec{B}_{\text{out}}\left(\vec{r}\right)$ for only the radial outflow! We will discuss the mathematics of this subtlety in more detail when we discuss magnetic monopoles, as in Chapter 6 of Jackson. For now, it suffices to closely adher to a physically complete current flow. To obtain a well-defined vector potential and magnetic field for the complete system near $z=+L_2$, we must consider both the wire and the radial outflow, in combination.

The simplest way to do this is to apply Ampere's law. Assuming there is a well-defined $\vec{B}$, for any surface S with boundary curve $C=\partial S$ the tangential integral of $\vec{B}$ around the boundary of the surface is equal to $\mu_0\times$ the flow of current through the surface, as follows from $\vec{\nabla}\wedge\vec{B}\left(\vec{r}\right)=\mu_0\vec{J}\left(\vec{r}\right)$ and Stokes' theorem.

$$\mu_{0}I_{\text{through }S}=\mu_{0}\int_{S}d^{2}\vec{a}\cdot\vec{J}\left(\vec{r}\right)=\int_{S}d^{2}\vec{a}\cdot\left(\vec{\nabla}\wedge\vec{B}\left(\vec{r}\right)\right)=\int_{C=\partial S}d\vec{s}\cdot\vec{B}\left(\vec{r}\right)$$

For simplicity, suppose that $L_1\to\infty$ and $L_2=0$, so we only need worry about outflow at one end of the wire. The symmetry of the current flow demands that $\vec{B}\left(\vec{r}\right)=B\left(\rho,z\right)\;\hat{e}_{\phi}$. If θ measures the usual polar angle from the positive z axis, by considering surfaces which are planar circular disks of radius ρ centered on the z axis and parallel to the xy plane, with circular boundaries at an angle θ from the z axis, Ampere's law gives

$$B\left(\rho,z\right)=\frac{\mu_{0}I}{4\pi\rho}\left(1-\frac{z}{\sqrt{z^{2}+\rho^{2}}}\right)$$

This follows because

$$I_{\text{through }S}=\int_{S}d^{2}\vec{a}\cdot\vec{J}\left(\vec{r}\right)=\frac{I}{4\pi}\int_{S}d\Omega=\frac{I}{2}\int_{0}^{\theta}\sin\theta\;d\theta=\frac{I}{2}\left(1-\cos\theta\right)$$

where $\cos\theta=z/\sqrt{z^2+\rho^2}$. Note that this is the correct net current through the plane of the disk, as measured to be positive in the $+z$ direction, even if the disk is below the origin and thus pierced by the wire.

The net result for $I_{\text{through }S}$ correctly includes both the jump discontinuity due to the current on the wire *and* the jump discontinuity due to the radial outflow from the origin, the latter due to the sign flip in the direction of the radially flowing current passing through the disk, as the disk is moved from positive to negative z. The net current through the disk is thus continuous as the disk is moved from positive to negative z. This is the same net current as would be obtained if we chose to take a surface S which is a spherical section (a spherical "cap") centered on the origin, extending from the z axis down to polar angle θ, and with the same circular boundary of radius ρ. In this case the current flux through the surface involves only the radial outflow since the wire does not pierce the surface at all for any $\theta<\pi$. So there is no current jump discontinuity through the spherical cap due to the wire. But neither is there a jump discontinuity through the cap due to the radial outflow.

In general, for a *fixed* boundary curve $C=\partial S$, the net current through the surface S will be the same for the complete physical system of the wire combined with the radial outflow, no matter how we vary the shape of the surface.

Now, let's exhibit a vector potential that gives

$$\vec{B}\left(\vec{r}\right)=\vec{\nabla}\wedge\vec{A}\left(\vec{r}\right)=\frac{\mu_{0}I}{4\pi\rho}\left(1-\frac{z}{\sqrt{z^{2}+\rho^{2}}}\right)\;\hat{e}_{\phi}$$

It is simple to find an acceptable $\vec{A}$ in cylindrical coordinates. With the somewhat odd-looking ansatz that $\vec{A}=A\left(\rho,z\right)\;\hat{e}_{z}$, we have

$$\vec{\nabla}\wedge\vec{A}\left(\vec{r}\right)=\left(-\frac{\partial}{\partial\rho}A\left(\rho,z\right)\right)\hat{e}_{\phi}$$

Thus we need to integrate

$$\frac{\partial}{\partial \rho} A(\rho, z) = \frac{\mu_0 I}{4\pi\rho} \left(-1 + \frac{z}{\sqrt{z^2 + \rho^2}} \right)$$

to find

$$A(\rho, z) = -\frac{\mu_0 I}{4\pi} \ln \left(\sqrt{z^2 + \rho^2} + z \right) + a(z)$$

where $a(z)$ is an arbitrary "gauge" function of integration that has no effect on the curl. Another way to write this is

$$A(\rho, z) = \frac{\mu_0 I}{4\pi} \left(-\ln \rho - \operatorname{arctanh} \frac{z}{\sqrt{z^2 + \rho^2}} \right) + a(z)$$

since

$$\ln \left(\sqrt{z^2 + \rho^2} + z \right) = \ln \rho + \operatorname{arctanh} \frac{z}{\sqrt{z^2 + \rho^2}}$$

Either way it is written, however, it is evident that this vector potential does *not* satisfy the Coulomb gauge condition.

$$0 \neq \vec{\nabla} \cdot \vec{A}(\vec{r}) = \frac{\partial}{\partial z} A(\rho, z) = -\frac{\mu_0 I}{4\pi} \frac{1}{\sqrt{z^2 + \rho^2}} + \frac{\partial}{\partial z} a(z)$$

On the other hand, we did not obtain this vector potential from the integral expression involving the current density, so there is no particular reason to expect the Coulomb gauge condition to hold!