PHY650 Notes on Bessel functions by T Curtright

$$
\begin{aligned}
J_\nu(x) &= \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\;\Gamma(1+\nu+j)} \left(\frac{x}{2}\right)^{2j} \\
&\underset{x\to 0}{\widetilde{\quad}} \frac{1}{\Gamma(1+\nu)} \left(\frac{x}{2}\right)^\nu \\
&\underset{x\gg 1}{\widetilde{\quad}} \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu}{2} - \frac{\pi}{4}\right) \\
&= \frac{1}{2\pi i^\nu} \int_0^{2\pi} d\phi\;\; \exp(ix\cos\phi - i\nu\phi) \qquad \text{integer } \nu \\
N_\nu(x) &= \frac{1}{\sin\nu\pi}\left(J_\nu(x)\cos\nu\pi - J_{-\nu}(x)\right) \quad \text{a.k.a.} \quad Y_\nu(x) \\
&\underset{x\to 0}{\widetilde{\quad}} -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu \quad \text{or} \quad \frac{2}{\pi}\left\{\ln\left(\frac{x}{2}\right) + \gamma_{Euler}\right\} \\
&\underset{x\gg 1}{\widetilde{\quad}} \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu}{2} - \frac{\pi}{4}\right) \\
\frac{2}{\pi x} &= W\left\{J_\nu(x), N_\nu(x)\right\}
\end{aligned}
$$

where

$$
\ln\Gamma(1+x) = -\ln(1+x) + (1-\gamma_{Euler})\,x + \sum_{n=2}^{\infty} (-1)^n \left[\zeta(n) - 1\right] x^n/n
$$

for $|x| < 2$ with $\gamma_{Euler} = 0.57721\ 56649\ \cdots$. For the modified functions,

$$
\begin{aligned}
I_\nu(x) &= \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{1}{j!\;\Gamma(1+\nu+j)} \left(\frac{x}{2}\right)^{2j} = i^{-\nu} J_\nu(ix) \\
&\underset{x\to 0}{\widetilde{\quad}} \frac{1}{\Gamma(1+\nu)} \left(\frac{x}{2}\right)^\nu \\
&\underset{x\gg 1}{\widetilde{\quad}} \sqrt{\frac{1}{2\pi x}} \exp(x) \\
K_\nu(x) &= \frac{\pi}{2\sin\nu\pi}\left(I_{-\nu}(x) - I_\nu(x)\right) \\
&\underset{x\to 0}{\widetilde{\quad}} \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu \quad \text{or} \quad -\left\{\ln\left(\frac{x}{2}\right) + \gamma_{Euler}\right\} \\
&\underset{x\gg 1}{\widetilde{\quad}} \sqrt{\frac{\pi}{2x}} \exp(-x) \\
\frac{1}{x} &= W\left\{K_\nu(x), I_\nu(x)\right\}
\end{aligned}
$$

Bessel's differential equation is

$$x\frac{d}{dx}\left(x\frac{d}{dx}J_\nu\right) = \left(\nu^2 - x^2\right) J_\nu$$

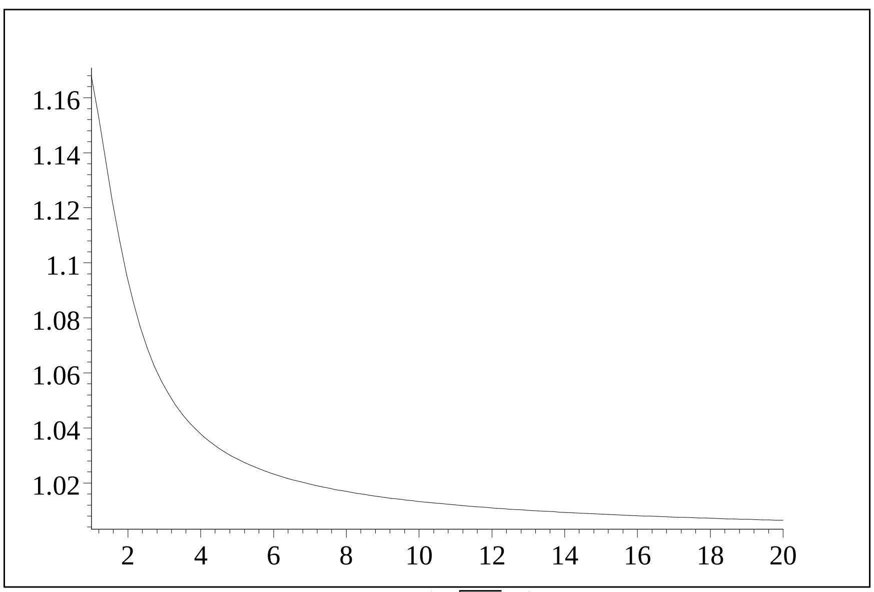
The modified differential equation is obtained by replacing $x \to ix$.

$$x\frac{d}{dx}\left(x\frac{d}{dx}I_\nu\right) = \left(\nu^2 + x^2\right) I_\nu$$

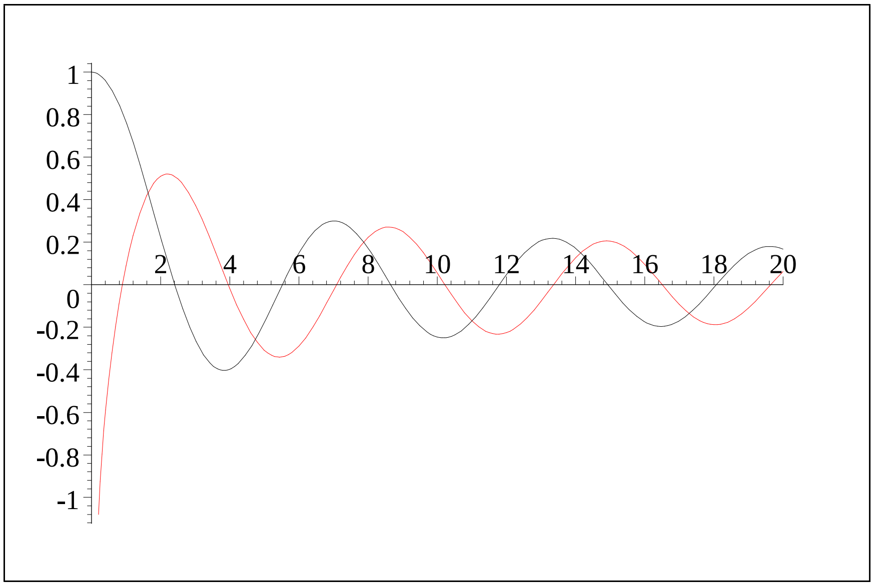
Alternatively, upon letting $s = \ln x$

$$\frac{d^2}{ds^2}I_\nu = \left(\nu^2 + e^{2s}\right) I_\nu$$

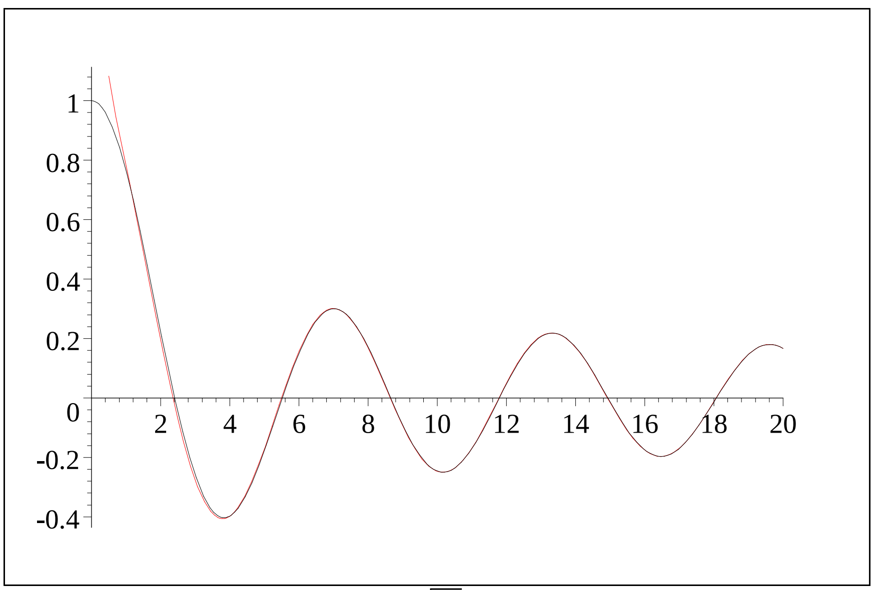
The asymptotic behavior of the solution is straightforward for this case, since we can easily discern dominant terms. Clearly, for large s, assuming that I and its derivatives are also large, we find $I_\nu \sim e^{e^s} = e^x$. Writing $I_\nu = f(s)\, e^{e^s}$ then transforms the differential equation into $f'' - \nu^2 f + e^s\left(2f' + f\right) = 0$, which for large s gives $2f' \sim -f$, and hence $\frac{d}{ds}\ln f \sim -1/2$. Thus $f \sim \exp(-s/2) = 1/\sqrt{x}$, and so $I_\nu \sim e^x/\sqrt{x}$. Now, there is no way to obtain the normalization factor $1/\sqrt{2\pi}$ just from the asymptotic behavior of the differential equation, since the equation is linear in f. It is necessary to connect the large x behavior to the small x behavior, where we have fixed the normalization through that of the above power series. We will be content to check the normalization numerically, for I_0, as given below.



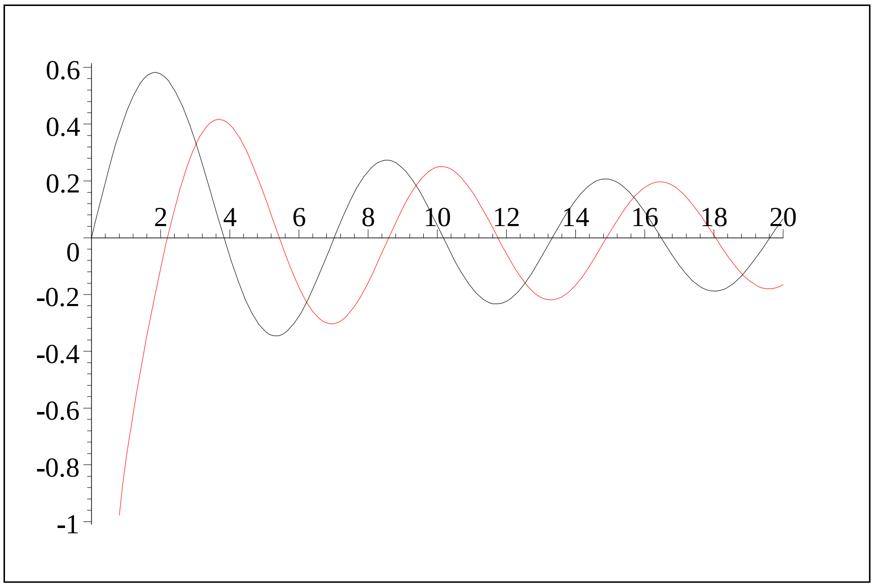
$I_0(x) / \left(\sqrt{\frac{1}{2\pi x}}e^x\right)$



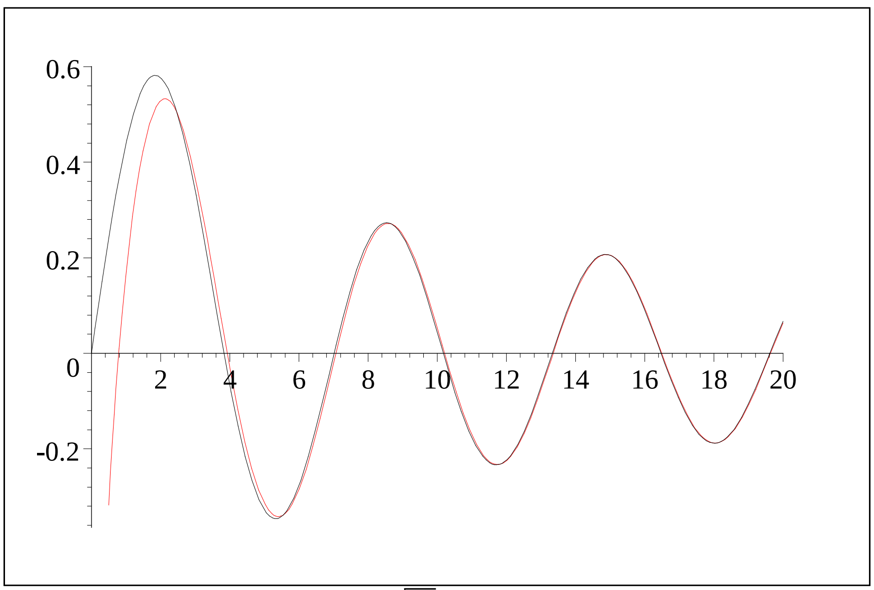
$J_0(x)$ and $Y_0(x)$ (or $N_0(x)$)



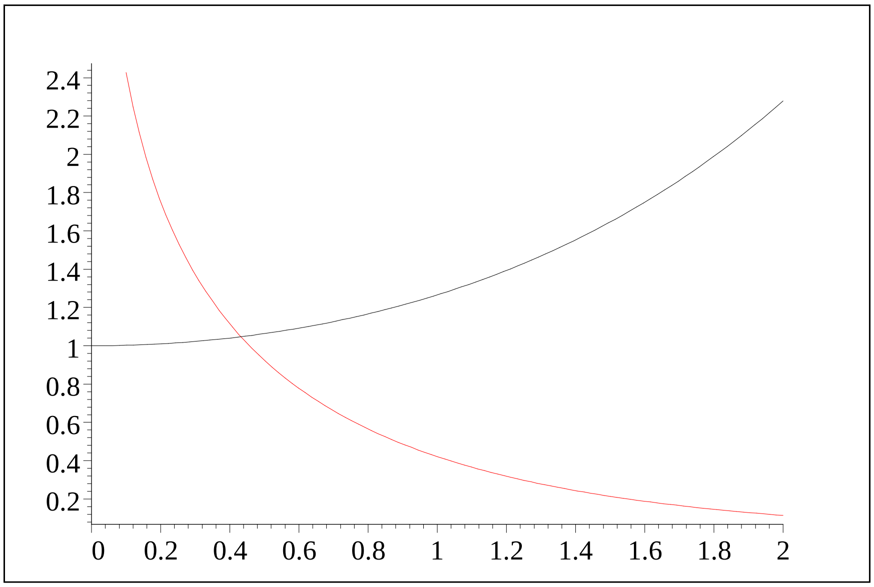
$J_0(x)$ and $\sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right)$



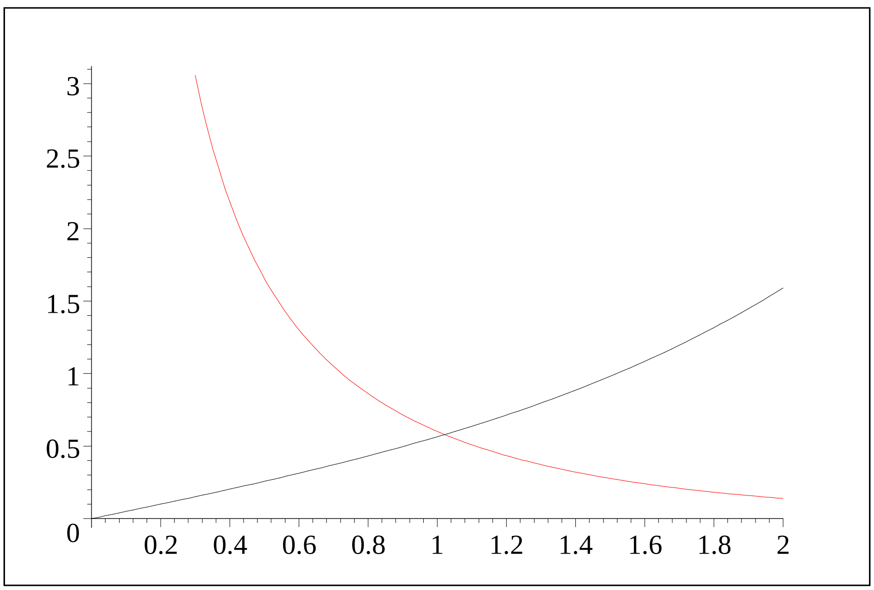
$J_1(x)$ and $Y_1(x)$ (or $N_1(x)$)



$J_1(x)$ and $\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{2}-\frac{\pi}{4}\right)$



$I_0(x)$ and $K_0(x)$



$I_1(x)$ and $K_1(x)$