"Think about neutrinos"

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Topics

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4 Conclusions and outlook
1 Base fermions and scalars in SO10

The 3 families in the (full) chiral basis form each a 16 spinor representation of SO10

Figure 1: Key questions → why 3 ? why SO10 ?
The left chiral notation shall be

\[( f_\gamma k ) \hat{\gamma}_F ; \hat{\gamma} = 1, 2 \ : \text{spin projection}\]

\( F = I, II, III \ : \text{family label}\)

\( k = 1, \cdots, 16 \ : \text{SO10 label}\)

Lets call the above extension of the standard model the ’minimal nu-extended SM’.
The lepton flavors can be filtered out of the 16 representations

\[
\begin{pmatrix}
\bullet & \bullet & \bullet & \nu & | & \mathcal{N} & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \ell & | & \hat{\ell} & \bullet & \bullet & \bullet
\end{pmatrix}^{\dot{\gamma}}_F = e, \mu, \tau
\]

(2)

\[
\begin{pmatrix}
\nu & \mathcal{N} \\
\ell & \hat{\ell}
\end{pmatrix}^{\dot{\gamma}}_F = e, \mu, \tau
\]

The three neutrino flavors $\mathcal{N}^F_F$ are to be identified with the entries in the figure as $\mathcal{N}^F_F = \overline{\mathcal{N}}^F_F$. 
The right-chiral base fields are then associated (by convention) to

\[
\begin{pmatrix}
  f^* \\
  k
\end{pmatrix}
_{F \alpha} = \varepsilon \alpha \gamma 
\begin{pmatrix}
  f \\
  \gamma
\end{pmatrix}
^*_{F}
\]

(3)

\[
(\varepsilon = i \sigma_2)_{\alpha \gamma} =
\begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix}
\]

The matrix \( \varepsilon \) is the symplectic \((\text{Sp}(1))\) unit, as implicit in Ettore Majorana’s original paper [1].

Yukawa interactions and mass terms

The doublet(s) of scalars are related to the 'tilt to the left'.

key question → why 'tilt to the left'? 

\[(\begin{array}{cc}
\nu & \mathcal{N} \\
\ell & \hat{\ell}
\end{array}) \quad F \leftrightarrow \quad \begin{array}{cc}
\varphi^0 & \Phi^+ \\
\varphi^- & \Phi^0
\end{array} = z\] 

The green entries in eq. (4) denote singlets under $SU_2^L$. 

The quantity $z$ is associated with the quaternionic or octonionic structure inherent to the $(2, 2)$ representation of $SU_2^L \otimes SU_2^R$ (outside of the electroweak gauge group $SU_2^L \otimes U_1^Y$) [2].
The minimal scalar doublet is associated with the reality restriction on \( z \), in components

\[
\Phi^+ = - (\varphi^-)^* , \quad \Phi^0 = (\varphi^0)^* 
\]

(5)

\[
\rightarrow z = \frac{1}{\sqrt{2}} \left( z^0 - i \bar{z} \bar{\sigma} \right) 
\]

The reality condition, incompatible with \( \text{susy} \), translates using the components \( z = (z^0, \bar{z}) \) to

(6)

\[
z^\kappa = (z^\kappa)^* ; \quad \kappa = 0, \ldots, 3 
\]

The Yukawa couplings are of the form (notwithstanding the quaternionic or octonionic structure of scalar doublets)

\[ \mathcal{H}_Y = \left[ (\varphi^0)^* , (\varphi^-)^* \right] \lambda_{F'F} \times \]

\[ \times \left\{ \varepsilon_{\dot{\gamma} \delta} \mathcal{N}_{F'}^{\dot{\delta}} \begin{bmatrix} \nu \dot{\gamma} \\ \ell \dot{\gamma} \end{bmatrix}_F \right\} + h.c. \]

(7)

\[ \mathcal{N}_{\dot{\gamma} F'} = \varepsilon_{\dot{\gamma} \delta} \mathcal{N}_{F'}^{\dot{\delta}} ; \varepsilon_{\dot{\gamma} \delta} = \varepsilon_{\gamma \delta} = \varepsilon_{\gamma \delta} \]

The only allowed Yukawa couplings by \( SU2_L \otimes U1_Y \) invariance are those in eq. (7), with arbitrary complex couplings \( \lambda_{F'F} \).
Spontaneous breaking of $SU_2_L \otimes U_1_\gamma$ through the vacuum expected value(s)

$$\langle \Omega \mid \begin{pmatrix} \varphi^0 & \Phi^+ \\ \varphi^- & \Phi^0 \end{pmatrix} (x) \mid \Omega \rangle =$$

$$= \langle z(x) \rangle = \begin{pmatrix} v_{ch} (v^u_{ch}) & 0 \\ 0 & v_{ch} (v^d_{ch}) \end{pmatrix}$$

$$v_{ch} = \frac{1}{\sqrt{2}} \left( \sqrt{2} G_F \right)^{-1/2} = 174.1 \text{ GeV}$$

independent of the space-time point $x$, [2], [3], [4]
induces a neutrino mass term through the Yukawa couplings $\lambda F' F$ in eq. (7) \(^a\). We use the shorthand

\(^a\) The implied parallelizable nature of $\langle z (x) \rangle$ is by far not trivial and relates in a wider context including triplet scalar representations to potential (nonabelian) monopoles and dyons.


\begin{equation}
F' \mathcal{N} \nu_F = \mathcal{N} \gamma F' \nu_{F'} = \nu F \mathcal{N}_{F'}
\end{equation}

\begin{equation}
\mu F' F = \nu_{\text{ch}} \lambda F' F
\end{equation}

\begin{equation}
\rightarrow \mathcal{H}_\mu = \mathcal{N} \mu_{F' F} \nu_F + \text{h.c.}
= \mathcal{N}^T \mu \nu + \text{h.c.} = \nu^T \mu^T \mathcal{N} + \text{h.c.}
\end{equation}

The matrix $\mu$ defined in eq. (9) is an arbitrary complex $3 \times 3$ matrix, completely analogous to the similarly induced mass matrices of charged fermions, i.e. charged leptons and quarks. In the setting of primary SO10 breakdown, a general (not symmetric) Yukawa coupling $\lambda_{F' F}$ implies the existence in the scalar sector of at least two irreducible representations $(16) \oplus (120)$.

key question $\rightarrow$ a 'drift' towards unnatural complexity? It becomes even worse including the heavy neutrino mass terms: 256 (complex) scalars.
2 'Mass from mixing’ in vacuo \[6\] - \[8\] or 'Seesaw’ = ’gigampfi’ \[9\] - \[12\]

The special feature, pertinent to (electrically neutral) neutrinos is, that the \(\nu\) extending degrees of freedom \(N\) are singlets under the whole SM gauge group

\[G_{SM} = SU3_c \otimes SU2_L \otimes U1_Y, \text{ in fact remain singlets under}\]

\(^{a}\) The documented discussions, but first for the general vectorlike situation, can be found in


and for 'our world, tilted to the left' in

[8] Peter Minkowski, ”\(\mu \rightarrow e\gamma\) at a rate of one out of 1-billion muon decays ?” , Phys.Lett.B67 (1977) 421.

\(^{b}\) Correct derivations were subsequently documented in →
the larger gauge group $SU_5 \supset G_{SM}$. This allows an arbitrary (Majorana-) mass term, involving however only the bilinears formed from two $\mathcal{N}$-s. In the present setup (minimal $\nu$-extended SM) the full neutrino mass term is thus of the form

[10] Tsutomu Yanagida, "Horizontal symmetry and masses of neutrinos", published in the Proceedings of the Workshop on the Baryon Number of the Universe and Unified Theories, O. Sawada and A. Sugamoto (eds.), Tsukuba, Japan, 13-14 Feb. 1979, and in (QCD161:W69:1979), and also in
\[ \mathcal{H} \mathcal{M} = \frac{1}{2} \begin{bmatrix} \nu & \mathcal{N} \end{bmatrix} \mathcal{M} \begin{bmatrix} \nu \\ \mathcal{N} \end{bmatrix} + \text{h.c.} \]

(10)

\[ \mathcal{M} = \begin{pmatrix} 0 & \mu^T \\ \mu & \mathcal{M} \end{pmatrix} ; \quad \mathcal{M} = \mathcal{M}^T \rightarrow \mathcal{M} = \mathcal{M}^T \]

Again within primary SO10 breakdown the full \( \mathcal{M} \) extends the scalar sector to the representations \((16) \oplus (120) \oplus (126)\).

key questions \( \rightarrow \) quo vadis ? is this a valid explanation of the 'tilt to the left' ? no , at least insufficient !

Especially the 0 entry needs explanation. It is an exclusive property of the minimal \( \nu \)-extension assumed here.
Since the 'active' flavors $\nu_F$ all carry $I_3 w = \frac{1}{2}$ terms of the form

$$\frac{1}{2} \ F' \nu \chi \ F' \ F \nu \ F = \frac{1}{2} \nu^T \chi \nu ; \ \chi = \chi^T$$

cannot arise as Lagrangean masses, except induced by an $I_w$-triplet of scalars, developing a vacuum expected value independent from the doublet(s).

from "The apprentice magician" by Goethe: 'The shadows I invoked, I am unable to get rid of now!'

'Seesaw'

The relative 'size' of $\mu$ and $M$ shall define the 'mass from mixing' situation and segregates 3 heavy neutrino flavors from the 3 light ones:

$$|\mu| \ll |M| \rightarrow$$
Figure 2: key questions → which is the scale of $M$? $O(10^{10})$ GeV → is there any evidence for this scale today? hardly! → and what about susy?
Diagonalization of $\mathcal{M}$

We shall use the generic expansion parameter

$$\vartheta = \left| \mu \right| / \left| M \right| \ll 1$$

and determine a unitary $6 \times 6$ matrix $U$ with the property

$$\mathcal{M} = U \mathcal{M}_{\text{diag}} U^T \rightarrow \mathcal{M}_{\text{diag}} =$$

$$\mathcal{M}_{\text{diag}} \left( m_1, m_2, m_3; M_1, M_2, M_3 \right)$$

$$0 \leq m_1 \leq \cdots \leq M_3, \ m_3 \ll M_1$$

\[\text{For a recursive treatment to all orders see:}\]

and

$$U = T U_0 \ ; \ T^{-1} \mathcal{M} T^{-1} T = \mathcal{M}_{\text{bl.diag.}} \rightarrow$$

$$= \begin{pmatrix}
\mathcal{M}_1 & 0 \\
0 & \mathcal{M}_2
\end{pmatrix} = U_0 \mathcal{M}_{\text{diag}} U_0^T$$

The matrix $T$ in eq. (14) describes the mixing of light and heavy flavors, determined from a $3 \times 3$ submatrix $t$.

$$T = \begin{pmatrix}
(1 + t t^\dagger)^{-1/2} & (1 + t t^\dagger)^{-1/2} t \\
-t^\dagger (1 + t t^\dagger)^{-1/2} & (1 + t^\dagger t)^{-1/2}
\end{pmatrix}$$
The matrix $t$ in eq. (15) is reduced to diagonal form through two unitary $3 \times 3$ matrices $u$ and $w$ \(^a\).

\[
t = u ( \tan a_{diag} ) w^{-1} ; \quad a_{diag} = a_{diag} ( a_1 , a_2 , a_3 )
\]

\[
0 \leq a_k \leq \pi / 2 ; \quad a_k \ll \pi / 2 \text{ for } \vartheta = | \mu | / | M | \ll 1
\]

(16)

$t$ is determined from the quadratic equation

\[
t = \mu^T M^{-1} - t \mu \bar{t} M^{-1}
\]

(17)

which can be solved recursively, setting

\[^a_{In \text{ eq. (16)} a_{diag} \text{ defines the three (real) heavy-light mixing angles } a_{1,2,3}, \text{ which without loss of generality can be chosen in the first quadrant, but which are small for } \vartheta = | \mu | / | M | \ll 1.\]
\[ t_{n+1} = \mu^T M^{-1} - t_n \mu \bar{t}_n M^{-1} \]

\[ t_0 = 0, \quad t_1 = \mu^T M^{-1}, \]

(18)

\[ t_2 = t_1 - \mu^T M^{-1} \mu \mu^\dagger \bar{M}^{-1} M^{-1} \]
\[ \ldots \]

\[ \lim_{n \to \infty} t_n = t \]

In order to control convergence we introduce the specific norms

\[ |\mu|^2 = \text{tr} \mu \mu^\dagger; \quad |M|^{-2} = \text{tr} M^{-1} M^{-1} \]

(19)

\[ \psi = |\mu| / |M| \ll 1 \]

\[ ^a \text{The sequence defined in eq.(18) is convergent for } \psi < 1. \]
the 'devil' is in the details

\( u, w \) in eq. (16) contain all 9 CP violating phases, pertaining to T. The above was intended to 'explain' why the (un)observed light neutrino masses are so much smaller than charged fermion ones.

key question → does it? wait.

\( t = u (\tan a_{\text{diag}}) w^{-1} \) defined in eq. (16) and its determining equation, repeated below

\[
t = \mu^T M^{-1} - t \mu \bar{t} M^{-1}
\]

ensure block diagonal form of \( \mathcal{M}_{bl.diag.} \).

\[
\mathcal{M}_{bl.diag.} = T^{-1} \mathcal{M} T^{-1} T; \quad \mathcal{M}_{bl.diag.} = \begin{pmatrix}
M_1 & 0 \\
0 & M_2
\end{pmatrix}
\]
\[(M_1, M_2) \text{ forming } M_{bl.diag.} \text{ defined in eq. (20) become}

\[M_1 = (1 + tt^\dagger)^{-1/2} \left[ -t\mu - \mu^T t t^T + t Mt^T \right] \times \]
\[\times (1 + tt^\dagger)^{-1/2T} \]
\[(21)\]
\[M_2 = (1 + t^\dagger t)^{-1/2} \times \left[ \mu \bar{t} + t^\dagger \mu^T + M \right] \times \]
\[\times (1 + t^\dagger t)^{-1/2T} \]

Comparing \(M_1\) with \(tM_2t^T\) we find
the relation \[ a \], \[[14]\]

(22) \[ \mathcal{M}_1 = - t \mathcal{M}_2 t^T \]

It follows from the assumptions detailed in footnote \(a\), that \(\text{Det} \ t \neq 0\) and hence the heavy-light mixing angles \(a_{1,2,3} > 0\) defined in eq. (16) are strictly bigger than 0. The lowest approximation, \(t \rightarrow t_1\) and \(\mathcal{M}_2 \rightarrow \mathcal{M}\), yields the first nontrivial approximation of the light neutrino mass matrix.

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\(^a\) In the scenario adopted here, we further assume \(\text{Det} \ M \neq 0\) and \(\text{Det} \ \mu \neq 0\). This leaves no room for light 'sterile' neutrinos, which would imply a nonminimal \(\nu\) — extension of the standard model. This would be mandatory, if the results of the LSND collaboration are correct.

\[[14]\] The LSND Collaboration (G.B. Mills for the collaboration), "Results on neutrinos from LSND", published in *Stanford 1998, Gravity from the Hubble length to the Planck length* 467-475.
in ‘second order mixing’

\[ M_1 \sim M_{1}^{(2)} = -\mu^T M^{-1} \mu \]

Remaining diagonalization of \( M_{bl.diag} \).

We go back to eq. (14) \( U = T U_0 \):

\( U_0 \) diagonalizes the \( 3 \times 3 \) blocks \( M_1, M_2 \)

\[
T^{-1} M T^{-1} T = M_{bl.diag}; \quad M_{bl.diag} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}
\]

\[ \text{(24)} \]
\[ U_0 = \begin{pmatrix} u_0 & 0 \\ 0 & v_0 \end{pmatrix} \sim U_0 I \]

\[ I = I_{\text{diag}} (\pm 1, \cdots, \pm 1) \]

(25)

\[ \mathcal{M}_1 = u_0 m_{\text{diag}} (m_1, m_2, m_3) u_0^T \]

\[ \mathcal{M}_2 = v_0 M_{\text{diag}} (M_1, M_2, M_3) v_0^T \]

\[ \mathcal{M}_1 = -t \mathcal{M}_2 t^T \]

\( T \) is constructed as a sequence (eq. 18), convergent for

\[ \vartheta = \left| \frac{\mu}{\left| M \right|} \right| < 1, \text{ as shown above, and thus unique.} \]

As a consequence of eq. (22) – \( t \) being determined (within \( T \)) –
\( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) and hence also \( u_0 \) and \( v_0 \) are not independent of each other. We shall keep as independent variables \( \mathcal{M}_1 \) and \( t \) (or equivalently \( T \)).

**Generic mixing and mass estimates**

1. Mixing

Here we introduce the arithmetic mean measure for \( 3 \times 3 \) matrices \( A \), not to be confused with the norms defined in eq.(19)

\begin{equation}
|A| = |Det A|^{1/3}
\end{equation}

Eq. (22) then implies
\[ \left| \mathcal{M}_1 \right| / \left| \mathcal{M}_2 \right| = \left| t \right|^2 \]

(27) \[ \left| \mathcal{M}_1 \right| = \left| m_{\text{diag}} \right| = \left( m_1 m_2 m_3 \right)^{1/3} \]

\[ \left| \mathcal{M}_2 \right| = \left| M_{\text{diag}} \right| = \left( M_1 M_2 M_3 \right)^{1/3} \]

Lets introduce the arithmetic mean of the light and heavy neutrino masses and the corresponding 'would be' masses if \( \mu \) and \( \mu^T \) would be the only parts of the full \( 6 \times 6 \) mass matrix \( \mathcal{M} \)

\[ \overline{m} = \left( m_1 m_2 m_3 \right)^{1/3} \quad ; \quad \overline{M} = \left( M_1 M_2 M_3 \right)^{1/3} \]

(28) \[ \mu = u_\mu \mu_{\text{diag}} \left( \mu_1, \mu_2, \mu_3 \right) v_\mu^{-1} \]

\[ \overline{\mu} = \left( \mu_1, \mu_2, \mu_3 \right)^{1/3} \]
Then beyond eq. (27) there is one more (exact) relation

\[ \hat{t} = \left( \tan a_1 \tan a_2 \tan a_3 \right)^{1/3} \]

| \mu |^2 = | M_1 | | M_2 | \rightarrow

\[ \frac{\overline{m}}{\overline{\mu}} = \hat{t}, \quad \frac{\overline{m}}{\overline{M}} = \hat{t}^2 \]

(29)

or equivalently

\[ \overline{m} = \hat{t} \overline{\mu}, \quad \overline{M} = \hat{t}^{-1} \overline{\mu} \]

seesaw (of type I)

\[ ^a \text{for MSSM inspired seesaw of type II realizations see e.g. [15]} \]
2. Mass

These estimates are based on the assumption that the scalar doublets (2) are part of a complex 10-representation of SO10 with the Yukawa couplings of the form

\[
\mathcal{H}_Y = \lambda F' F \begin{pmatrix} 16 & 16 & 10 \\ B & A & D \end{pmatrix} \ F' \bar{f} \ B \bar{f} \ A \ F + h.c.
\]

(30)

\[
\rightarrow \lambda F' F = \lambda F F'
\]

It follows that at the unification scale we have a

---

a In order to obtain a general (not a symmetric) heavy-light mass matrix $\mu$ a combination of SO10 representations $(120) \oplus (16)$ is needed, which however would (could) 'destroy' the mass relation in eq. (29). Key question $\rightarrow$ is this relevant? Estimate shall be estimate.
We shall use the relation at a scale near 100 GeV

\begin{equation}
\mu = \mu^T = \mu_u
\end{equation}

(31)

\begin{equation}
\mu \sim \frac{1}{3} (\mu_u)
\end{equation}

(32)

The factor $\frac{1}{3}$ accounts for the color rescaling reducing the (colored) up-quark mass matrix from the unification scale down to 100 GeV.

It follows using the definitions in eq. (28) and the quark masses $m_u \sim 3$ MeV, $m_c \sim 1$ GeV and $m_t \sim 180$ GeV

\begin{equation}
\overline{\mu_u} = \left( m_u m_c m_t \right)^{1/3} \sim 0.81 \text{ GeV} \rightarrow
\end{equation}

(33)

\begin{equation}
\overline{\mu} \sim 0.27 \text{ GeV}
\end{equation}
Further let's approximate the mass square differences, obtained from the combined neutrino oscillation experiments, by

\[ \Delta m_{12}^2 \sim 10^{-4} \text{ eV}^2 \]

\[ \Delta m_{23}^2 \sim 2.5 \times 10^{-2} \text{ eV}^2, \]

(34)

Finally 'pour fixer les idées' I set the lowest light neutrino mass \( \sim 1 \text{ meV} \) and assume hierarchical \((123)\) light masses. This implies

\[ m_1 \sim 1 \text{ meV}, \quad m_2 \sim 10 \text{ meV}, \quad m_3 \sim 50 \text{ meV} \]

(35)

\[ \rightarrow \quad \overline{m} \sim 8 \text{ meV} \]
It follows from eq. (29)

\[ \hat{t} = \frac{m}{\mu} \sim 2.9 \times 10^{-11} ; \quad \overline{M} = \frac{\mu}{\hat{t}} \sim 0.9 \times 10^{10} \text{ GeV} \]

(36)
\[ \hat{t}^2 \sim 0.9 \times 10^{-21} \]

Light neutrino masses are indeed small.\(^a\)

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\(^a\) Key questions → is susy bringing down in 'small steps' the B-L protecting mass scale \( \overline{M} = \sim 0.9 \times 10^7 \text{ TeV} \) to \( 1 \text{ TeV} \) ? , or is \( \overline{M} = \sim 0.9 \times 10^7 \text{ TeV} \) in view of seesaw type II too small ? , \( \mu \rightarrow e \gamma \) at a rate of ?
3 Main neutrino oscillations in vacuo

'low' energy release in production
'low' energy 'gain' in detection

In short ...

3a) Production of neutrinos much below the energy scale of heavy flavors

First let's consider production processes, whereby the heavy neutrino flavor states are 'blocked', i.e. inhibited by energy conservation.

Then we can eliminate the heavy flavors

\[
\nu \rightarrow \overline{U}_{11} \hat{\nu} , \quad \nu^\dagger \rightarrow \hat{\nu}^\dagger U_{11}^T
\]
The production amplitude shall be characterized by a definite time $t_P$, in the system of 'preference', yielding – in amplitude – a state we denote by

\begin{equation}
| \Psi_\nu (t_P) \rangle
\end{equation}

We use the local asymptotic fields associated with the mass eigenfields $\hat{\psi}$ to represent this state by means of the full Majorana extended asymptotic fields as

\begin{equation}
| \Psi_\nu (t_P) \rangle = \int d^3 p (2\pi)^{-3} \Phi_F (t_P, \vec{p}, h) | \vec{p}, h, F \rangle
\end{equation}

\begin{equation}
| \vec{p}, h, F \rangle = A_F^* (\vec{p}, h) | \Omega \rangle
\end{equation}
In eq. (39) the 'neutrino wave packet' described by the functions \( \Phi_F (t_P, \vec{p}) \) is meant to define the full production process to a sufficient approximation in amplitude.

\( A_F (\vec{p}_F, h) \) are the nonrelativistically normalized absorption operators for the mode characterized by the spinor \( u^\Gamma (\vec{p}_F, h) \). They satisfy the anticommutation relations

\[
\{ A_F (\vec{p}, h), A_{F'}^\ast (\vec{p}', h') \} = (2\pi)^3 \delta^3 (\vec{p} - \vec{p}') \delta_{hh'} \delta_{FF'}
\]

(40)

\[
E_F^p = \sqrt{\vec{p}^2 + m_F^2}
\]
The normalization is

\[ \langle \Psi_\nu (t_P) | \Psi_\nu (t_P) \rangle = \int d^3 p (2\pi)^{-3} \times \]

\[ \times \Phi^*_F (t_P, \vec{p}, h) \Phi_F (t_P, \vec{p}, h) \]

(41)

We now return to the ’low’ production process of neutrinos, which shall derive from the charged current

(42) \[ j_\mu^+ = \bar{\nu}_F \gamma_\mu^L \ell_F \]

yielding the state \[ | \Psi_\nu (t_P) \rangle \] described above, through the
production amplitude

\[ A_{\nu \, P} = A_{\nu \, P}^F = \left\langle \Psi_\nu (t \, P) ; \cdots \left| \bar{\nu}_F \gamma^L \ell_F \cdots \right| \text{in} \right\rangle \]

\[ = \left\langle \Psi_\nu^{(F)} (t \, P) \left| \hat{\nu}^{\dagger} \right| \Omega \right\rangle (U_{11})_{FF'} \cdots \]

\[ = \int d^3 \, p \left( 2\pi \right)^{-3} (U_{11})_{FF'} \Phi^{* (F)}_{F'} (t \, P, \vec{p}, h) \cdots \]

\[ = \int d^3 \, p \left( 2\pi \right)^{-3} (U_{11})_{FF'} \Phi^{* (F)}_{F'} (t \, P, \vec{p}, h) \cdots \]

no sum over \( F \)

(43)
The neutrino flavor $F$ is determined by tagging the charged (anti)lepton $\ell_F$, antilepton if produced together with $\nu_F$. The structure of $A_{\nu_P}^F$ is for $\Delta \bar{p}^2 \gg |\Delta m_{AB}^2|$:

$$A_{\nu_P}^F = (U_{11})_{FF'} \Phi^{*}_{F'}(t_P, \bar{p}, h) \cdots$$

$$\rightarrow \Phi_{F'}(t_P, \bar{p}, h) = \left\{ (\overline{U}_{11})^{-1} \right\}_{F'F} \Xi_F(t_P, \bar{p}, h)$$

$$U_{11} = (1 + tt^\dagger)^{-1/2} u_0$$

$$\left\{ (\overline{U}_{11})^{-1} \right\} = u_0^T (1 + \bar{t} t T)^{1/2} = u_0^T (1 + O(\hat{t}^2))$$

$$= u_0^T (1 + O(\sim 0.9 \times 10^{-21}))$$

(44)
We will set $U_{11} \to u_0$ from here on, but retain, that $U_{11}$ is only exactly unitary if $t = 0$, i.e. for vanishing light neutrino masses. Then $\Phi^{(F)}_{F'}$ in eq. (39) becomes

\begin{equation}
\Phi^{(F)}_{F'}(t_P, \vec{p}, h) = (u_0)_{FF'} \Xi_F(t_P, \vec{p}, h)
\end{equation}

The functions $\Xi_F$ in eq. (45) describe the mixing independent wave packets induced by the production of $\nu_F$. The latter is assumed to inherit an intrinsic spread in momenta, with mean momentum $\langle \vec{p} \rangle$. The latter could well depend on the lepton flavor $F$, characterizing the production process. The main helicity component – for neutrino production – is $h = -1$, although of course both helicities are actually $a$.

\[a\text{It is often assumed 'à priori', that } U_{11} \text{ is necessarily unitary, as is indeed the case for the CKM matrix. This exact unitarity is impossible in the case of 'mass from mixing'.}\]
produced from the charged current. The ratio of ’wrong’ to right helicity production is

\[ \frac{\Xi_F(t_P, \vec{p}, -1)}{\Xi_F(t_P, \vec{p}, 1)} \approx \left( \frac{O\left( \frac{m^2}{<\vec{p}>^2} \right)}{O\left( \frac{m^2}{m^2_{\mu}} \right) \text{ for } \pi \text{ decay}} \right)^2 \]

(46)

The ’wrong’ helicity component would dominantly appear as antineutrino in subsequent detection. \footnote{The neutrino \( \to \) antineutrino transition between production and detection would prove overall lepton flavor (B-L) violation. key questions \( \to \) why is this violation so much suppressed, or so well protected \( \to \) is this a consequence of the ’tilt to the left’ ?}

Henceforth we set \( \Xi_F(t_P, \vec{p}, 1) \to 0 \).
3b) From production to detection

We take the state \( \left| \Psi^{(F)}_{\nu} \left( t_P \right) \right\rangle \), defined in eqs. (38, 39 and 41)

\[
\left| \Psi^{(F)}_{\nu} \left( t_P \right) \right\rangle \sim \left( u_0 \right)_{F F'} \int d^3 p \left( 2\pi \right)^{-3} \times
\]

\[
\times \Xi_F \left( t_P, \vec{p}, -1 \right) \left| \vec{p}, -1, F' \right\rangle
\]

(47)

no sum on F

and choose configuration space coordinates \( \rightarrow \)
\[ t_P = 0, \quad \vec{x}_P = \langle \vec{x} \rangle = 0 \]

(48) \[ \langle \vec{x} \rangle = \int d^3 p \left( \frac{2\pi}{p} \right)^{-3} \Xi^*_F \left( t_P, \vec{p}, -1 \right) \times \]
\[ \times \left( i \partial_{\vec{p}} \right) \Xi_F \left( t_P, \vec{p}, -1 \right) \]

\( t_P, \vec{x}_P \) defined in eq. (48) designate the 'exact' production time \( (t_P) \) and average production point \( (\vec{x}_P) \) respectively.

Next we propagate the state \[ \left| \Psi^{(F)}_{\nu} \left( t_P \right) \right\rangle \] to an arbitrary time \( t \), which will be identified with the detection time \( t_D \).

\[ t \rightarrow t_D \rightarrow \]
\[ \left| \Psi_{\nu}^{(F)} (t) \right\rangle = \left( u_0 \right)_{FF'} \int d^3 p \left( \frac{2\pi}{\hbar} \right)^{-3} \times \]

\[ \times \Xi_F (0, \vec{p}, -1) \left| \vec{p}, -1, F' \right\rangle \exp (-i E_{\vec{p}} \cdot t) \]

\[ x = (t, \langle \vec{x} (t) \rangle) \rightarrow x_D (-x \cdot p) \]

3c) Detection of neutrinos much below the energy scale of heavy flavors

We proceed in a way analogous to the production amplitude \( \mathcal{A}_{\nu P}^{F} \) in eq.(43) and consider the detection amplitude \( \mathcal{A}_{\nu D}^{G} \) yielding in the final – detected – state a charged lepton with flavor G from the charged current (contragredient to \( j_{\mu}^{+} \) in eq.(42))
As for production, characterised by the 'neutrino wave packet in production' defined by the functions $\Phi_F(t_P, \vec{p})$ in eq. (39), the detection process gives rise to 'neutrino wave packets in detection' characterized by a sum over states we shall denote as

$$\sum_Z \left| Z^{(G)}_\nu \right\rangle \langle Z^{(G)}_\nu \right|$$
\[ \mathcal{A}_{\nu D}^{G \leftrightarrow F} = \sum_{Z} \langle \text{out}, G | \bar{\ell}_G \cdots | \text{in} ; \text{det} \rangle \times \left| Z_{\nu}^{(G)} \right\rangle \times \]
\[ \times \left\langle \left| Z_{\nu}^{(G)} \right\rangle | \Psi_{\nu}^{(F)}(x) \right\rangle \]
\[ \left| Z_{\nu}^{(G)} \right\rangle \rightarrow (u_0)_{GG'}, \int d^3 p \ (2\pi)^{-3} \times \]
\[ \times \Delta_G \left( t_D, \bar{p}^\ast, -1 \right) | \bar{p}^\ast, -1, G' \rangle \]
\[ (51) \]

In eq. (51) the scattering (in) state \( | \text{in} ; \text{det} \rangle \) denotes the incoming state within the detector, e.g. a neutron for the detecting reaction \( \nu + n \rightarrow \ell^- + p \).
Each of the states $\left| Z_{\nu}^{(G)} \right\rangle$ is determined by an associated wave function $\Delta_{G}(t_{D}, \vec{P}, -1)$, in analogy with $\left| \Psi_{\nu}^{(F)}(x) \right\rangle$, (eq. 49) characterising the x-propagated production process. The structure of both 'neutrino wave packets', a collection (density matrix) for detection and exactly one (wave function) for production

$$\left| Z_{\nu}^{(G)} \right\rangle, \left| \Psi_{\nu}^{(F)}(x) \right\rangle \rightarrow$$
\[ \left| Z_{\nu}^{(G)} \right\rangle \rightarrow \]

\[ \rightarrow (u_0)_{GG'} \int d^3 p (2\pi)^{-3} \Delta_{G}(t_D, \vec{p}', -1) \left| \vec{p}', -1, G' \right\rangle \]

\[ \left| \Psi_{\nu}^{(F)}(x) \right\rangle \rightarrow (u_0)_{FF'} \int d^3 p (2\pi)^{-3} \times \]

\[ \times \Xi_{F}(0, \vec{p}', -1) \left| \vec{p}', -1, F' \right\rangle \exp(-i E_{\vec{p}'}^F t) \]

(52)

The neutrino wave packet projection in detection

\[ P_{\text{det}}^{G} = \sum_{Z} \left| Z_{\nu}^{(G)} \right\rangle \left\langle Z_{\nu}^{(G)} \right| \]
is necessary in principle, to guarantee the geometric acceptance of the detector.  

Nevertheless we will assume in the following that the spread in position and/or momentum of the production process is precise enough, so that $P_{det}^G$ can be replaced by a complete sum over one particle states

$$P_{det}^G \rightarrow \left( u_0 \right)_{GG}, \int d^3 p \left( \frac{2\pi}{\hbar} \right)^{-3} \times$$

$$\times \left| \vec{p}, -1, G' \right> \left< \vec{p}, -1, G' \right| \times \left( u_0^* \right)_{GG},$$

(53)

An eventually necessary geometrical acceptance correction can be applied 'classically', as is done for beam transport over short beam lines conventionally.

\[\text{a}\] The essential feature of quantum mechanics, that not both position and momentum can be determined together has led to a 'propagation of errors', which shall not be documented here.
3d) **Summary**

1) The remaining parts of the amplitudes $A^{FP}_{\nu P}$ for production (eq. 43) and $A^{G}_{\nu D}$ for detection (eqs. 50-51) can be approximated neglecting light neutrino masses.

2) Coherent and decoherent properties of 'low-low' neutrino oscillations $G \leftarrow F$ in vacuo and for the main helicity components, are contained, using the approximation defined in eq. (52) – in amplitude – in the scalar products

\[
\begin{align*}
(\begin{array}{c}
\nu^* \\
0
\end{array})_{GF'}, \left\langle \vec{p}, -1 G' \right| \Psi^{(F)}_{\nu}(x) \right) &= \\
&= e^{-i E_{p'}^{F} t} \Xi_{F}(0, \vec{p}, -1) \times (\begin{array}{c}
\nu^* \\
0
\end{array})_{GF'} (\begin{array}{c}
\nu \\
0
\end{array})_{FF'}
\end{align*}
\]
We can proceed from eq. (54) in various steps of approximations, and here choose the following path:

3) **Neglect of direct retardation**

In short for \( |\vec{p}| \gg mF \)

(55)

\[
t \rightarrow L \rightarrow (\langle E \rangle / \langle p \rangle) L
\]

in general

where \( L \) is the mean distance between detection and production.

4) **\( G \leftarrow F \) transition probabilities**

In short – to lowest order in \( m^2_A \) – and for a Gaussian wave function we obtain
\[ \langle \vec{p} \rangle = (0, 0, k) \quad \text{and} \quad \xi_{FG;A} = (u^*_0) G_A (u_0) F_A \]

\[ w(G \leftarrow F; L) \rightarrow \sum_{BA} \xi_{FG;B}^* \xi_{FG;A} \times \]

\[ \times e^{i \frac{1}{2k} (\Delta m^2)_{BA} L} \tilde{q}_{BA}^F (L) \]

The fluctuation distribution \( \tilde{q}_{BA}^F (L) \) becomes.
\( \tilde{\varrho}^F_{BA}(L) = \left[ \left( 1 - 2i \sum \frac{2}{3} \Theta_{BA} \right)^{-1/2} \times \left( 1 + i \sum \frac{2}{\perp} \Theta_{BA} \right)^{-1} \right] \times \right.

\left( -\frac{1}{2} \frac{\sum \frac{2}{3} (\Theta_{BA})^2}{1 - 2i \sum \frac{2}{3} \Theta_{BA}} \right) \right)

(57)

\( \Theta_{BA} = \frac{1}{2k} \left( \Delta m^2 \right)_{BA} L \)

\( \sum_{3} = \Delta p_{\parallel} / \langle p_{\parallel} \rangle ; \sum_{\perp} = \Delta p_{\perp} / \langle p_{\parallel} \rangle \)

Letting \( |\Theta_{BA}| \) assume large value we find asymptotically
\[ | \tilde{q}^F_{BA} (L) | \to \left( \frac{1}{\sqrt{2} \sum_3 \sum_2} \right) | \Theta_{BA} |^{-3/2} \times \]

\[ \times \exp \left( -\frac{1}{8 \sum_3^2} \right) \text{ for } | \Theta_{BA} | \to \infty \]

\[ (\sum_3)^{-1} = \frac{\langle p_\parallel \rangle}{\Delta p_\parallel} \gg 1 \]

The \( | \Theta_{BA} |^{-3/2} \) behaviour shows the normal decay of a wave packet, whereas decoherence sets in – for \( \frac{\langle p_\parallel \rangle}{\Delta p_\parallel} \gg 1 - \)
for

\[ \frac{L}{L_{AB\ osci}} = \frac{\langle p_\parallel \rangle}{\Delta p_\parallel} \]

\[ L_{AB\ osci} = 2 \frac{\langle p_\parallel \rangle}{\mid \Delta m^2 \mid_{BA}} ; \quad L_{AB\ osci} = \frac{L_{AB\ osci}}{2\pi} \]

\[ L_{AB\ osci} = 2.48 \text{ km} \frac{\langle p_\parallel \rangle (\text{GeV})}{\mid \Delta m^2 \mid_{BA} (\text{eV}^2)} \]

\[ \frac{L_{AB\ osci}}{\lambda (\langle p_\parallel \rangle)} = 2 \frac{\langle p_\parallel \rangle^2}{\mid \Delta m^2 \mid_{BA}} \quad (\rightarrow 2 \times 10^{18})^a \]

(59)

\[ ^a_{\text{For} \mid \Delta m^2 \mid = 10^{-4} \text{ eV}^2 \text{ and } p_\parallel = 10 \text{ MeV.}} \]
4 Conclusions and outlook

1) Neutrino properties are only to a very small extent open (up to the present) to deductions from oscillation measurements.

2) Notwithstanding this, a significant and admirable experimental effort paired with theoretical analysis has revealed the main two oscillation modes. The matter effect due to Mikheev, Smirnov and Wolfenstein demonstrates another clear form of quantum coherence, over length scales of the solar radius.

3) Key questions remain to be resolved: are all (ungauged or gauged) global charge-like quantum numbers violated? (B-L), B, L, individual lepton flavors.
4) SO10 served fine (together with susy or without it) to guide ideas, but a genuine unification is as remote as the scales and nature of heavy neutrino flavors, to name only these.

Yet, the quest for unification is wide open.