Scale and special conformal transformations Given a conserved and symmetric energy-momentum tensor $\theta_{\mu\nu}$,

$$\partial^\mu \theta_{\mu\nu} = 0 \ , \quad \theta_{\mu\nu} = \theta_{\nu\mu} \tag{1}$$

we consider the divergences of various currents made by taking first and second moments of $\theta_{\mu\nu}$. Thus

$$M_{\mu\nu\lambda} \equiv \theta_{\mu\nu} x_\lambda - \theta_{\mu\lambda} x_\nu \ , \quad S_\mu \equiv \theta_{\mu\nu} x^\nu \ , \quad K_\mu{}^\nu \equiv \theta_{\mu\lambda} \left(x^2 \eta^{\lambda\nu} - 2x^\lambda x^\nu \right) \tag{2}$$

with $x^2 = x^\mu x_\mu$. Note the first of these is antisymmetric in the 2nd and 3rd indices, while the last is neither symmetric nor antisymmetric. It is in fact

$$K_{\mu\nu} = x^2 \theta_{\mu\nu} - 2 S_\mu x_\nu \tag{3}$$

In any case, we use (1) to obtain

$$\begin{aligned}
\partial^\mu M_{\mu\nu\lambda} &= \left(\partial^\mu \theta_{\mu\nu}\right) x_\lambda + \theta_{\lambda\nu} - \left(\partial^\mu \theta_{\mu\lambda}\right) x_\nu - \theta_{\nu\lambda} \\
&= 0 &\text{(4a)} \\
\partial^\mu S_\mu &= \left(\partial^\mu \theta_{\mu\nu}\right) x^\nu + \theta_\mu{}^\mu \\
&= \theta_\mu^\mu &\text{(4b)} \\
\partial^\mu K_\mu{}^\nu &= \left(\partial^\mu \theta_{\mu\lambda}\right) \left(x^2 \eta^{\lambda\nu} - 2x^\lambda x^\nu \right) + \theta_{\mu\lambda} \partial^\mu \left(x^2 \eta^{\lambda\nu} - 2x^\lambda x^\nu \right) \\
&= \theta_{\mu\lambda} \left(2x^\mu \eta^{\lambda\nu} - 2\eta^{\mu\lambda} x^\nu - 2x^\lambda \eta^{\mu\nu} \right) \\
&= -2\theta_\mu^\mu x^\nu &\text{(4c)}
\end{aligned}$$

Therefore, the first of these (Lorentz transformation currents) are always conserved, while the other two types of currents are conserved if and only if

$$\theta_\mu^\mu = 0 \tag{5}$$

For theories with such *traceless* and symmetric, conserved energy-momentum tensors, there are more conserved quantities than generate the four types of translations

$$P_\mu = \int \theta_{0\mu} d^3x \tag{6}$$

and the six types of Lorentz transformations

$$J_{\nu\lambda} = -J_{\lambda\nu} = \int \left(\theta_{0\nu} x_\lambda - \theta_{0\lambda} x_\nu \right) d^3x \tag{7}$$

Namely, there are 5=1+4 additional scale and special conformal transformations generated by

$$\begin{aligned}
S &= \int S_0 d^3x = \int \theta_{0\nu} x^\nu d^3x \\
K_\nu &= \int K_{0\nu} d^3x = \int \left(\theta_{0\nu} x^2 - 2\theta_{0\lambda} x^\lambda x_\nu \right) d^3x &\text{(8)}
\end{aligned}$$

So altogether there are at least 15 conserved generators when

$$\partial^\mu \theta_{\mu\nu} = 0 \ , \quad \theta_{\mu\nu} = \theta_{\nu\mu} \ , \quad \theta_\mu^\mu = 0 \ . \tag{9}$$

These 15 generators exponentiate to give the conformal group, $SO(4,2)$, containing the Poincaré (also known as the inhomogeneous Lorentz) and Lorentz groups, $SO(4,2) \supset ISO(3,1) \supset SO(3,1)$.