

PHY612 Problems and Exercises, Spring 2009

Problem 1 Consider a forced harmonic oscillator described by

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 - f(t) x$$

where the driving term switches on at $t = 0$. That is to say, $f(t) \equiv 0$ for all $t \leq 0$, while for $t > 0$ the driving force is a reasonably well-behaved but otherwise arbitrary function. Suppose that the system is in the usual oscillator ground state $|n = 0\rangle$ for $t \leq 0$. Compute $\langle n | 0, t \rangle$, and $p(n, t) = |\langle n | 0, t \rangle|^2$, the amplitude and probability that the force causes the oscillator to make a transition into the n th energy eigenstate of the ($f = 0$) oscillator. Check that your answer gives $\sum_{n=0}^{\infty} p(n, t) = 1$ for arbitrary $f(t)$. For constant $f(t > 0)$, what is $p(n, t)$, and what is the traditional name for this probability distribution? Also compute $\langle n \rangle = \sum_{n=0}^{\infty} n p(n, t)$, and compare your answer to the average energy gained by the forced oscillator during the time t .

Exercise 1 Restore the mass and coupling parameters to the generator for the classical canonical transformation: $H = p^2 + x \rightarrow -P$. That is to say, find a generator $F(x, X)$, with $p = \partial F(x, X) / \partial x$ and $P = -\partial F(x, X) / \partial X$, that produces the transformation $H = \frac{1}{2m} p^2 - f x = -\gamma P$, for constant m, f , and γ .

Exercise 2 For $t > 0$ the propagator for the Hamiltonian $H = \frac{1}{2m} p^2 - f x$, in position representation with $p = -i\hbar \frac{\partial}{\partial x}$, is

$$K(x, X; t) = \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left(i \frac{m}{2\hbar} \frac{(x - X)^2}{t} + i \frac{f}{2\hbar} (x + X) t - i \frac{1}{24m\hbar} f^2 t^3\right)$$

Here, the parameter f is a constant. Verify that this solves the time-dependent Schrödinger equation in the position representation.

Exercise 3 Find the classical action for a trajectory connecting x_1 at time t_1 to x_2 at time t_2 for a particle whose dynamics are given by $L = \frac{m}{2} \left(\frac{dx}{dt}\right)^2 + f(t) x$. Note that the driving term, i.e. $f(t)$, may have an explicit time dependence. You may assume that $f(t)$ is differentiable, and otherwise reasonably well behaved.

Exercise 4 Consider the Heisenberg picture Hamiltonian $H(t) = \frac{1}{2m} p(t)^2 - f(t) x(t)$, where $x(t)$ and $p(t)$ are canonically conjugate operators at time t . Show that

$$H(t) = H(0) - \int_0^t x(\tau) \frac{df(\tau)}{d\tau} d\tau$$

Exercise 5 Show that

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n = \frac{t^n}{n!}$$

More generally, for the case $[F(t_1), F(t_2)] = 0$, for all $0 \leq t_1, t_2 \leq t$, show that

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n F(\tau_1) F(\tau_2) \cdots F(\tau_n) = \frac{1}{n!} \left(\int_0^t d\tau F(\tau) \right)^n$$

Exercise 6 Specifically, for any well-behaved numerically valued function, $f(\tau)$, establish the integral identities

$$\left(\int_{t_1}^{t_2} f(\tau) d\tau \right)^2 = 2 \int_{t_1}^{t_2} d\tau_1 f(\tau_1) \int_{t_1}^{\tau_1} d\tau_2 f(\tau_2)$$

$$\left(\int_{t_1}^{t_2} (\tau - t_1) f(\tau) d\tau \right)^2 = 2 \int_{t_1}^{t_2} d\tau_1 (\tau_1 - t_1) f(\tau_1) \int_{t_1}^{\tau_1} d\tau_2 (\tau_2 - t_1) f(\tau_2)$$

Should it be unclear how to proceed, you might begin by comparing the LHS and RHS for the special case $f(\tau) = a\tau^m + b\tau^n$.

Exercise 7 Show that the fluctuation prefactor \mathcal{F} occurring in the propagator for a linearly driven particle, as discussed in class, satisfies the first order equation

$$i\hbar \frac{d}{dt_2} \mathcal{F}(t_2, t_1) = \frac{1}{2m} \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (\tau - t_1) f(\tau) d\tau \right)^2 \mathcal{F}(t_2, t_1)$$

Exercise 8 For the Schrödinger picture Hamiltonian $H_S = \frac{1}{2m} p^2 - f(t) x$, use the results of the previous two exercises, and provide all the details, to show complete cancelation of the x and p independent coefficients that arise in the time-dependent Schrödinger equation obeyed by the operator-valued evolution transformation, $U(t_2, t_1)$, as given in class.

Exercise 9 Find the classical action for a SHO particle trajectory connecting x_1 at time t_1 to x_2 at time t_2 where the dynamics are given by $L = \frac{m}{2} \left(\frac{dx}{dt} \right)^2 - \frac{1}{2} m \omega^2 x^2$, and where m and ω are constants.

Exercise 10 Fill in all the algebraic steps to establish the \mathbf{x} and \mathbf{p} operator identity discussed in lecture:

$$\exp \left(-\frac{i}{\hbar} t \left(\frac{1}{2m} \mathbf{p}^2 + \frac{1}{2} m \omega^2 \mathbf{x}^2 \right) \right)$$

$$= \exp \left(-\frac{im\omega}{2\hbar} \tan \left(\frac{\omega t}{2} \right) \mathbf{x}^2 \right) \exp \left(-\frac{i}{2m\hbar\omega} \sin(\omega t) \mathbf{p}^2 \right) \exp \left(-\frac{im\omega}{2\hbar} \tan \left(\frac{\omega t}{2} \right) \mathbf{x}^2 \right)$$

Exercise 11 Show that the propagator for the SHO follows by taking the $\langle x_1 | \cdots | x_2 \rangle$ matrix element of this last operator identity, and explicitly check that your result for the propagator obeys the time-dependent SHO Schrödinger equation in the position representation.

Exercise 12 Generalize the operator identity for the SHO, as given above, to the case of an oscillator moving in D dimensions. Allow both m and ω to be “anisotropic.” What is the corresponding expression for the propagator?

Exercise 13 Explicitly evaluate the “time-sliced path integral” for the free particle to obtain, in the limit of an infinite number of equally spaced slices, the result for the free particle propagator.

$$K(x, X; t) = \sqrt{\frac{m}{2\pi i \hbar t}} \exp \left(i \frac{m}{2\hbar} \frac{(x - X)^2}{t} \right)$$

Generalize this result to describe a free particle moving in D dimensions.

Exercise 14 Perform the intermediate p_k ($k = 1, \dots, N + 1$) momentum integrations in the time-sliced phase-space form of the functional integral for the propagator, for the intermediate Hamiltonians

$$H(x_k, p_k; t_k) = \frac{1}{2m} p_k^2 + A(x_k; t_k) p_k + V(x_k; t_k) ,$$

to obtain the intermediate Lagrangians $L(x_k, x_{k-1}; t_k)$ that appear in the remaining time-sliced path integral.

Exercise 15 Continue the integral relation

$$\frac{1}{\pi^{s/2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty t^{\frac{1}{2}s-1} \Theta(t) dt ,$$

valid for $\text{Re } s > 1$, to show the relation

$$\frac{1}{\pi^{s/2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(t^{\frac{1}{2}s-1} + t^{-\frac{1}{2}s-\frac{1}{2}}\right) \Theta(t) dt ,$$

valid for all $s \in \mathbb{C}$. Here we have used a “theta function” defined by the sum $\Theta(t) \equiv \sum_{n=1}^\infty e^{-n^2 \pi t}$. (Hint: Use the “Jacobi transformation” identity $1+2\Theta(t) = \frac{1}{\sqrt{t}} (1+2\Theta(\frac{1}{t}))$.)

Exercise 16 The 3-dimensional SHO radial propagator for fixed angular momentum ℓ is given for $t' > t$ by

$$K_\ell(r', t'; r, t) = \frac{m\omega\sqrt{rr'}}{i\hbar \sin(\omega(t' - t))} I_{\ell+\frac{1}{2}}\left(\frac{m\omega r r'}{i\hbar \sin(\omega(t' - t))}\right) \exp\left(\frac{im\omega}{2\hbar} (r^2 + r'^2) \frac{\cos(\omega(t' - t))}{\sin(\omega(t' - t))}\right)$$

Show that this obeys the time-dependent, radial Schrödinger equation with potential $V_{\text{effective}}(r) = \frac{\hbar^2 \ell(\ell+1)}{2mr^2} + \frac{1}{2}m\omega^2 r^2$.

Exercise 17 Expand the modified Bessel function in the previous result as a sum of Laguerre polynomial bilinears, hence obtain the radial wave functions for the oscillator.

Exercise 18 Let $r = R^2$ to show $\frac{1}{r} \frac{d^2}{dr^2} (r f(r)) = \frac{1}{4R^{7/2}} \frac{d^2}{dR^2} (R^{3/2} f(R^2)) - \frac{3}{16R^4} f(R^2)$.

Exercise 19 Change variables from r, t to $R = \sqrt{r}, s = \frac{t}{4r}$ in the time-dependent, radial Schrödinger equation for the Coulomb potential. What is the resulting partial differential equation?

Exercise 20 Use the projection of the 3-sphere onto Euclidean 3-space, for which $d\Omega = \left(\frac{2}{1+r^2}\right)^3 d^3r$, to compute the total solid angle on the 3-sphere: $\Omega = \int d\Omega$.

Exercise 21 For the harmonic oscillator described by $H(x, p) = \frac{1}{2}(x^2 + p^2)$, solve the eigenvalue equation $H \star f_n = (n + \frac{1}{2}) \hbar f_n = f_n \star H$ to obtain

$$f_n(x, p) = \frac{(-1)^n}{\pi \hbar} L_n\left(\frac{2}{\hbar}(x^2 + p^2)\right) e^{-(x^2 + p^2)/\hbar}$$

where the Laguerre polynomials are given by $L_n(z) = \frac{1}{n!} e^z \frac{d^n}{dz^n} (z^n e^{-z})$.

Exercise 22 For a pure state Wigner function normalized such that $\int dx dp f(x, p) = 1$, prove that

$$-\frac{1}{\pi\hbar} \leq f(x, p) \leq +\frac{1}{\pi\hbar} .$$

Exercise 23 Use the integral form of the \star product to show

$$\exp\left(-\frac{a}{\hbar}(x^2 + p^2)\right) \star \exp\left(-\frac{b}{\hbar}(x^2 + p^2)\right) = \frac{1}{1+ab} \exp\left(-\frac{a+b}{(1+ab)\hbar}(x^2 + p^2)\right)$$

Hence two Gaussians \star -commute with one another if they are centered on the same point (here, the origin of phase space). Note that these are ordinary exponentials, not \star -exponentials. Were the latter involved, we would have just $\exp_{\star}\left(-\frac{a}{\hbar}(x^2 + p^2)\right) \star \exp_{\star}\left(-\frac{b}{\hbar}(x^2 + p^2)\right) = \exp_{\star}\left(-\frac{a+b}{\hbar}(x^2 + p^2)\right)$.

Exercise 24 For the SHO with $H(x, p) = \frac{1}{2}(x^2 + p^2)$,

$$\exp_{\star}(itH(x, p)/\hbar) = \frac{1}{\cos(t/2)} \exp\left(\frac{i}{\hbar}(x^2 + p^2) \tan(t/2)\right) .$$

Verify this by showing that the explicit function on the RHS is a solution of

$$H(x, p) \star \exp_{\star}(itH(x, p)/\hbar) = -i\hbar \frac{\partial}{\partial t} \exp_{\star}(itH(x, p)/\hbar) = \exp_{\star}(itH(x, p)/\hbar) \star H(x, p)$$

with initial condition $\exp_{\star}(itH(x, p)/\hbar)|_{t=0} = 1$.

Exercise 25 Another way to present time evolution in phase space is to compute the “Wigner transform” of the propagator in coordinate space ($\hbar \equiv 1$ here)

$$\mathbb{K}(x, p; X, P; t) = \frac{1}{2\pi} \int dy dY K\left(x + \frac{1}{2}y, X + \frac{1}{2}Y; t\right) e^{-ipy+iPY} K^*\left(x - \frac{1}{2}y, X - \frac{1}{2}Y; t\right) ,$$

so that the Wigner function evolves as

$$f(x, p; t) = \int dXdP \mathbb{K}(x, p; X, P; t) f(X, P; 0) .$$

For the SHO of the previous problem, where

$$K(x, X; t) = \frac{1}{\sqrt{2\pi i \sin t}} \exp\left(\frac{i}{2}(x^2 + X^2) \cot t\right) - \frac{ixX}{\sin t} ,$$

evaluate the Wigner transform to show that the phase space propagator is simply Dirac deltas for the classical motion.

$$\mathbb{K}(x, p; X, P; t) = \delta(x \cos t - p \sin t - X) \delta(x \sin t + p \cos t - P) .$$

Exercise 26 For the linear potential problem, where $H = p^2 + x$, obtain in closed form the Wigner energy eigenfunctions $f_E(x, p)$, such that $H \star f_E(x, p) = E f_E(x, p) = f_E(x, p) \star H$. Note the energy spectrum is continuous, $-\infty < \sigma(H) < +\infty$, so the Wigner functions will have a continuum normalization. From the explicit form for $f_E(x, p)$, or otherwise, show that $f_{E_1}(x, p) \star f_{E_2}(x, p) = 0$ if $E_1 \neq E_2$.

Exercise 27 What is the phase-space propagator $\mathbb{K}(x, p; X, P; t)$ for the linear potential problem?

Exercise 28 Express the star exponential $\exp_\star(\alpha(p^2 + x))$ in terms of the ordinary exponential $\exp(\beta(p^2 + x))$. What is the relation between α and β ?

Exercise 29 Solve $i\partial_t U(t) = H \star U(t) = U(t) \star H$ for the linear potential problem, with $U(0) = 1$, in terms of ordinary functions.

Exercise 30 Show that

$$\begin{aligned} \frac{1}{1 + \tau\partial_\sigma} \cos(a\sigma) &= \frac{1}{1 + (a\tau)^2} (\cos(a\sigma) + a\tau \sin(a\sigma)) , \\ \frac{1}{1 + \tau\partial_\sigma} \sin(a\sigma) &= \frac{1}{1 + (a\tau)^2} (\sin(a\sigma) - a\tau \cos(a\sigma)) . \end{aligned}$$

Exercise 31 Work out the “twisted” field phase-space propagator $\mathcal{K}[x(\sigma), p(\sigma); X(\sigma), P(\sigma); \tau]$ for the linear potential problem, where the field interacts with an external source that may vary with position, i.e. the interaction energy is $V = \int f(\sigma) x(\sigma) d\sigma$. Compare your result to the conventional field propagator $K[x(\sigma), p(\sigma); X(\sigma), P(\sigma); t]$ for the same problem. What is the relation between τ and t ?

Exercise 32 For the $su(2)$ algebra of the classical ($m = 1 = \omega$) oscillator, as realized by

$$J_x = \frac{1}{2}(p_x p_y + xy) , \quad J_y = \frac{1}{2}(p_y^2 + y^2 - p_x^2 - x^2) , \quad J_z = \frac{1}{2}(xp_y - yp_x) ,$$

show that

$$J_x^2 + J_y^2 + J_z^2 = \frac{1}{16}(p_y^2 + p_x^2 + x^2 + y^2)^2$$

Hence establish that Poisson bracket time-evolution on the phase space is equivalent to Nambu 4-bracket evolution, in the following sense.

$$\{f, J_x^2 + J_y^2 + J_z^2\}_{PB} = 2 \times \{f, J_x, J_y, J_z\}_{4NB} .$$

The classical 4-bracket here is defined by the Jacobian

$$\{A, B, C, D\}_{4NB} = \frac{\partial(A, B, C, D)}{\partial(x, p_x, y, p_y)} \equiv \det \begin{pmatrix} \partial A/\partial x & \partial B/\partial x & \partial C/\partial x & \partial D/\partial x \\ \partial A/\partial p_x & \partial B/\partial p_x & \partial C/\partial p_x & \partial D/\partial p_x \\ \partial A/\partial y & \partial B/\partial y & \partial C/\partial y & \partial D/\partial y \\ \partial A/\partial p_y & \partial B/\partial p_y & \partial C/\partial p_y & \partial D/\partial p_y \end{pmatrix} .$$