

PHY650 Final Exam
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This is an open textbook exam, for Chapters 1 - 7, Jackson.

You may use without proof results given in the text of Jackson,
unless the problem statement specifically asks you to derive those results.

If you use results from the text, you must indicate precisely
what is being used and where it is in the text (for example,
give the formula number).

You may also use your notes. But **no other references** may be used!

You may **not** discuss the exam with anyone except Professor Curtright.

Good luck!

Name:

Student ID Number:

Problem 1 *Warm-up with a simple potential problem.*

Consider a *non-conducting* hemi-spherical surface, $0 \leq \theta \leq \pi/2$, of radius R covered with a fixed surface charge of density $\sigma = \frac{Q}{R^2} P_2(\cos \theta)$. Here, θ is the usual polar angle measured from the top of the hemisphere, and P_2 is a Legendre polynomial.

(a) Compute the electric multipole moments of this charge distribution. What is the total charge on the hemisphere?

Now place a smaller *conducting* sphere, of radius $a < R$, within and concentric to the hemisphere. Both sphere and hemi-sphere are centered on the same point. The total charge on this conducting sphere is q .

(b) When static equilibrium is reached, what is the charge density on the surface of the conducting sphere?

(c) What is the electric potential at the top ($\theta = 0$, $r = a$) of the smaller sphere?

Problem 2 *Now a little math interlude.*

In class we discussed in some detail an integral representation for Bessel functions of integer index.

$$J_n(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz \sin \theta} e^{-in\theta} d\theta$$

However, this is not useful for non-integer indices. But there is at least one elegant contour integral representation that works for all ν and all z , namely

$$J_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{2\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \exp\left(t - \frac{z^2}{4t}\right) dt$$

The contour begins at $-\infty$, with $\arg t = -\pi$, proceeds below the real t axis towards the origin, loops in the positive, counterclockwise sense around the origin (hence the $(0+)$ notation), and then continues above the real t axis back to $-\infty$, with $\arg t = +\pi$.

(a) Through integration by parts, show directly that this integral representation obeys the recursion relations expected for J_ν , and that it is indeed a solution of Bessel's equation.

$$\left(z \frac{d}{dz}\right)^2 J_\nu(z) = (\nu^2 - z^2) J_\nu(z)$$

(b) Show that the contour integral gives the usual infinite series solution for $J_\nu(z)$ in powers of z , for all ν .

Hint: Hankel noted and proved $\frac{1}{\Gamma(\nu)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} t^{-\nu} e^t dt$. Perhaps you should too.

Problem 3 *Just to see if you were paying attention in class.*

Point charges $\pm q(t)$ are at locations $z = \pm a$, with a straight filamentary wire connecting the two charges. The wire carries a current $i(t) = I + Wt$ in the z -direction such that the the point charges vary with time according to

$$q(t) = Q + It + \frac{1}{2}Wt^2$$

with constant Q , I , and W .

- (a) Determine $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$. Are there any “retardation effects” in your final answer?
- (b) Determine the electromagnetic energy flow by computing the Poynting vector $\vec{S}(\vec{r}, t)$.

Problem 4 *Filling in some details for your notes.*

In the radiation gauge, with $\vec{\nabla} \cdot \vec{A} = 0$, we may Fourier analyze an arbitrary vector potential wave *in vacuum* as

$$\vec{A}(\vec{r}, t) = \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3} \left(\vec{e}_{\lambda}(\vec{k}) a_{\lambda}(\vec{k}) e^{i\vec{k} \cdot \vec{r} - i\omega(\vec{k})t} + c.c. \right)$$

where the frequencies are given by $\omega(\vec{k}) = kc$. The sum over polarizations can be taken over linear transverse polarizations

$$\vec{e}_{1,2}(\vec{k}), \quad \vec{k} \cdot \vec{e}_{1,2}(\vec{k}) = 0$$

where the unit vectors $\{\vec{e}_1, \vec{e}_2, \hat{k}\}$ form a RH coordinate basis. Alternatively, the sum \sum_{λ} may be expressed in terms of circular polarizations

$$\vec{e}_{\pm}(\vec{k}) = \frac{1}{\sqrt{2}} \left(\vec{e}_1(\vec{k}) \pm i\vec{e}_2(\vec{k}) \right), \quad \vec{k} \cdot \vec{e}_{\pm}(\vec{k}) = 0$$

(a) For such a wave, derive an expression for the long-time-averaged Poynting vector as an integral over \vec{k} of a bilinear in the Fourier coefficients, $a_{\lambda}(\vec{k})$.

(b) Likewise, derive an expression for the long-time-averaged energy density.

Problem 5 *More on Gauss and Green functions for electrostatics.*

A static charge source distribution ρ vanishes outside a finite region V bounded by a closed surface S , with $\rho = 0$ for points on S . Let \vec{r} denote points outside V and away from the surface S , let \vec{s} denote points on the surface S , and let \vec{t} denote points inside V . Let G be the usual unbounded-space Green function,

$$G(\vec{r}) = \frac{1}{4\pi|\vec{r}|} \quad , \quad \nabla^2 G(\vec{r}) = -\delta^3(\vec{r})$$

Let G_D be the Dirichlet Green function appropriate to V ,

$$\nabla_t^2 G_D(\vec{t}, \vec{t}') = -\delta^3(\vec{t} - \vec{t}') \quad , \quad G_D(\vec{t}, \vec{s}) \Big|_{\vec{s} \in S} = 0$$

and let G_N be the *usual* Neumann Green function appropriate to V , with

$$\nabla_t^2 G_N(\vec{t}, \vec{t}') = -\delta^3(\vec{t} - \vec{t}') \quad , \quad \hat{n}(\vec{s}) \cdot \nabla_s G_N(\vec{t}, \vec{s}) \Big|_{\vec{s} \in S} = \frac{-1}{A}$$

where A is the area of S .

(a) By applying Green's theorem, show that

$$G(\vec{r} - \vec{t}) = - \iint_S d^2s \, G(\vec{r} - \vec{s}) \, \hat{n}(\vec{s}) \cdot \nabla_s G_D(\vec{s}, \vec{t})$$

where $\hat{n}(\vec{s})$ is the outward normal at the surface point \vec{s} .

(b) Use this result to show Gauss' *other* theorem: One can define a surface charge σ on S such that for all points outside V the potential due to σ alone would be the same as that due to ρ . Express σ in terms of ρ and G_D .

(c) Similarly, show that

$$\Phi(\vec{r}) = \frac{Q}{A} \iint_S d^2s \, G(\vec{r} - \vec{s}) + \iint_S d^2s \, \hat{n}(\vec{s}) \cdot \nabla_s G(\vec{r} - \vec{s}) \iint_V d^3t \, G_N(\vec{t}, \vec{s}) \rho(\vec{t})$$

and interpret this result in terms of a surface charge and a surface distribution of dipoles. Here

$Q = \iiint_V d^3t \, \rho(\vec{t})$ is the total charge within V .

Problem 6 *Static magnetic field problem.*

Jackson, 3rd Edition, Problem 5.17