PHY651 Final Exam
Thomas Curtright
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This is an open textbook exam (Jackson, 3rd Edition only).

If you use results from the text, you must indicate precisely what is being used and where it is in the text (for example, give the formula number). But no credit will be given for quoting a Jackson result *if* you are asked in the exam problem to provide the derivation of that result. (Obviously!)

You may also consult any lecture notes taken in class this semester.

You may **not** discuss the exam with anyone except Professor Curtright.

No credit will be given **unless** you work in SI units.

Good luck!

"On my honor, I have neither received nor given aid on this exam."

Signature:

Print Name:

Student ID Number:

Problem 1 *A modicum of understanding* (Make sure you use SI units throughout this problem.)

For monochromatic electric and magnetic fields

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\, e^{-i\omega t} \ , \quad \vec{B}(\vec{r},t) = \vec{B}(\vec{r})\, e^{-i\omega t}$$

the relations between the fields and potentials are

$$\vec{E}(\vec{r}) = -\nabla\Phi(\vec{r}) + i\omega\vec{A}(\vec{r}) \ , \quad \vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r})$$

To save some writing here, we have suppressed the frequency label on all fields, so $\Phi(\vec{r})$ is actually $\Phi_\omega(\vec{r})$, etc.

(a) Starting from Maxwell's equations for monochromatic fields *and* current/charge sources, derive the inhomogeneous Helmholtz equations obeyed by potentials $\vec{A}(\vec{r})$ and $\Phi(\vec{r})$ when subject to the Lorenz gauge condition: $\nabla \cdot \vec{A}(\vec{r}) = \frac{i\omega}{c^2}\Phi(\vec{r})$.

(b) What are the integral solutions for the potentials that behave as $\sim \frac{1}{r}\exp(ikr)$ when all sources vanish rapidly for large r?

(c) From the integral solution for $\vec{A}(\vec{r})$ obtain the corresponding integral solution for $\vec{B}(\vec{r})$.

(d) From your answer to (c), Fourier re-compose the fields and sources to obtain the "retarded" time-dependent integral solution for general $\vec{B}(\vec{r},t)$.

Problem 2 *A light vector workout* (Make sure you use SI units throughout this problem.)

A multipole field is known to be purely $l, m = 1, 0$ with magnetic field given in a sourceless region by

$$\overrightarrow{H}\left(\overrightarrow{r}\right) = H \; j_1\left(kr\right) \; \overrightarrow{Y}_{1,0}\left(\widehat{r}\right)$$

where H is a constant. Recall that $j_1\left(s\right) = \frac{\sin s}{s^2} - \frac{\cos s}{s}$ and $Y_{1,0}\left(\theta, \phi\right) = \sqrt{\frac{3}{4\pi}} \cos\theta$.

(a) Is this an electric multipole (TM), or a magnetic multipole (TE) field? Why?

(b) What are the various components of $\overrightarrow{H}\left(\overrightarrow{r}\right)$?

(c) What are the components of the corresponding electric field in the same sourceless region?

Problem 3 *A poor student's polaroid* (Make sure you use SI units throughout this problem.)

A single point charge q is constrained to move along an infinite straight line "wire" that lies in the xz-plane. The wire passes through the origin and makes an angle ϑ with respect to the z-axis. The charge q can move freely along the wire, but it cannot leave the wire. There are no other charges on the wire except q.

A monochromatic plane wave of frequency ω moves in the $+y$ direction and is incident on the charge-wire system, with the charge initially at the origin. The plane wave has electric polarization $E\,\widehat{e}_{in}$. The charge accelerates in response to the incident wave's electric field, and moves non-relativistically with $v \ll c$, but always subject to the constraint that it stays on the wire.

(a) For large r, determine the electric field for the scattered radiation $\overrightarrow{E}_{sc}\left(\overrightarrow{r}, t\right)$ in this situation. (You may neglect radiative reaction effects, if you wish!)

(b) Determine the polarized differential scattering cross-sections $d\sigma\left(\widehat{e}_{sc}, \widehat{e}_{in}\right)/d\Omega$ in this situation for the various allowed polarizations.

(c) What is the total scattering cross section σ_{un} for "unpolarized" incident waves?

Problem 4

Under a special conformal transformation of coordinates,

$$x^\mu \to \frac{x^\mu + x^2 s^\mu}{1 + 2x \cdot s + x^2 s^2} \tag{1}$$

where $x^2 \equiv x^\nu x_\nu$, $s^2 \equiv s^\nu s_\nu$, and $x \cdot s = x^\nu s_\nu$. This can be written as

$$e^{s^\nu K_\nu} \; x^\mu = \frac{x^\mu + x^2 s^\mu}{1 + 2x \cdot s + x^2 s^2} \tag{2}$$

where K_ν is an explicitly x-dependent, first-order differential operator.

(a) What is K_ν here? Justify your answer.

(b) Explicitly check your answer by verifying that when $e^{s^\nu K_\nu}$ acts on x^μ it gives $\dfrac{x^\mu + x^2 s^\mu}{1 + 2x \cdot s + x^2 s^2}$ up to and including all terms of second order in s^α.

(c) Prove that your result for K_ν, when exponentiated as in (2), gives the right-hand-side of (2) to all orders in s^α.

Problem 5 (Make sure you use SI units throughout this problem.)

The covariant wave equation satisfied by the field strength, in the presence of a given 4-current $J_\mu(x)$, is

$$\partial^\lambda \partial_\lambda F_{\mu\nu}(x) = \mu_0 \left(\partial_\mu J_\nu(x) - \partial_\nu J_\mu(x)\right) \tag{3}$$

(a) Assuming the given current and its partial derivatives are highly localized in both space and time, construct the "time-delayed" solution of this equation as an integral involving the current and the covariant form of the retarded Green's function. You may assume that there are no "incoming" free fields unrelated to J_μ.

(b) For a point particle of mass m and charge q with a given trajectory, $X^\mu(\tau)$, parameterized by the particle's proper time τ, show that your solution of part (a) becomes

$$F_{\mu\nu}(x) = \frac{q}{4\pi\varepsilon_0} \frac{1}{(x-X)_\alpha V^\alpha} \frac{d}{d\tau} \left(\frac{(x-X)_\mu V_\nu - (x-X)_\nu V_\mu}{(x-X)_\beta V^\beta} \right) \Bigg|_{\tau=\tau_0} \tag{4}$$

where $V^\alpha(\tau) = dX^\alpha(\tau)/d\tau$ is the particle's 4-velocity, and where all terms in the expression are evaluated (*after* differentiating) at the retarded proper time τ_0 determined as the delayed solution of the light-cone condition $0 = (x - X(\tau_0))^\lambda (x - X(\tau_0))_\lambda$. It may be helpful for this part of the problem to note that the 4-current for the point particle can be conveniently written as

$$J_\mu(x) = cq \int d\tau \; V_\mu(\tau) \; \delta^4(x - X(\tau)) \tag{5}$$

(c) By considering the components of $F_{\mu\nu}$ (i.e. $\overrightarrow{E}$ and $\overrightarrow{B}$) and of J_μ (i.e. ρ and $\overrightarrow{J}$), show that your answer to part (a) is the same as Jefimenko's results, as given by (6.55) and (6.56) in Jackson.

Problem 6 (Make sure you use SI units throughout this problem.)

Larmor's formula for the instantaneous total radiated power (obtained by integrating the flux over all directions) emitted from an accelerating non-relativistic point particle of mass m and charge q is

$$\mathcal{P} = \frac{2}{3}\left(\frac{q^2}{4\pi\varepsilon_0 m^2 c^3}\right)\left|\frac{d\overrightarrow{p}}{dt}\right|^2 \tag{6}$$

where $\overrightarrow{p} = m\overrightarrow{v}$ is the particle's non-relativistic momentum.

(a) *Given* that this power is the non-relativistic limit of a Lorentz scalar, obtain the *unique*, manifestly Lorentz invariant form for $\mathcal{P}$ in terms of the particle's 4-momentum p_μ and/or the proper-time derivatives $dp_\mu/d\tau$. Consider and explain how to rule out all other invariants constructed from p_μ and/or $dp_\mu/d\tau$.

(b) Show in complete detail that your Lorentz invariant answer for $\mathcal{P}$ is exactly the same as Liénard's formula, valid for relativistic motion:

$$\mathcal{P} = \frac{2}{3}\left(\frac{q^2}{4\pi\varepsilon_0 c^3}\right)\left\{\gamma^4\left|\overrightarrow{a}\right|^2 + \gamma^6\left(\overrightarrow{a}\cdot\overrightarrow{v}/c\right)^2\right\} \tag{7}$$

where $\overrightarrow{v}$ is the particle's instantaneous velocity, $\overrightarrow{a} = d\overrightarrow{v}/dt$, and $\gamma = 1/\sqrt{1-v^2/c^2}$ is the particle's Lorentz factor.

Problem 7 (Make sure you use SI units throughout this problem.)

For a point particle of mass m, Newton's equations

$$m\frac{d^2\overrightarrow{x}}{dt^2} = \overrightarrow{F} \tag{8}$$

can be generalized to the Lorentz covariant form

$$\frac{dp^\mu}{d\tau} = F^\mu \tag{9}$$

provided the force $\overrightarrow{F}$ has been carefully incorporated into a 4-vector F^μ, whose spatial part reduces to $\overrightarrow{F}$ in the non-relativistic limit, and which in general satisfies the constraint

$$p_\mu F^\mu = 0 \tag{10}$$

This constraint is required by the invariant mass condition, $p_\mu p^\mu = m^2c^2$, from which it follows that $p_\mu dp^\mu/d\tau = 0$.

With the same provisos, the Abraham-Lorentz equation with both radiation reaction and non-electromagnetic force $\overrightarrow{F}$

$$m\frac{d^2\overrightarrow{x}}{dt^2} = m\tau\frac{d^3\overrightarrow{x}}{dt^3} + \overrightarrow{F}\ , \quad \tau \equiv \frac{2}{3mc^3}\frac{q^2}{4\pi\varepsilon_0} \tag{11}$$

can be generalized to Lorentz covariant form.

(a) Do so, for the case of $\overrightarrow{F} = 0$, and verify that your result is consistent with $p_\mu dp^\mu/d\tau = 0$.

(b) For the case $\overrightarrow{F} \neq 0$, specialize to motion in only one spatial dimension, say the z direction, with a non-electromagnetic force given as some arbitrary function of the proper time, $\widehat{z}\cdot\overrightarrow{F} = f(\tau)$, and write a Lorentz covariant form of (11). Solve for $p_z(\tau)$ in terms of $p_z(0)$ for arbitrary $f(\tau)$.

Problem 8 (Make sure you use SI units throughout this problem.)

Assume a linear, homogeneous, isotropic, causal connection between $\vec{E}$ and $\vec{D}$, as in Jackson §7.10, especially (7.105). Similarly for $\vec{B}$ and $\vec{H}$, but for simplicity suppose the permeability is that of a vacuum, so just assume $\vec{B} = \mu_0 \vec{H}$. Fourier transform the space and time dependence for all quantities, defining

$$\vec{E}\left(\vec{x}, t\right) = \frac{1}{(2\pi)^2} \int d\omega \int d^3k \; \vec{E}\left(\vec{k}, \omega\right) \; e^{i\vec{k}\cdot\vec{x} - i\omega t} \tag{12}$$

etc. (We will distinguish space-time fields from their Fourier transforms by indicating the independent variables on which they depend, but we will use the same symbols for both types of fields.) Then the linear relations between fields become (see (7.103) in Jackson)

$$\vec{D}\left(\vec{k}, \omega\right) = \varepsilon(\omega)\, \vec{E}\left(\vec{k}, \omega\right) \; , \quad \vec{B}\left(\vec{k}, \omega\right) = \mu_0 \vec{H}\left(\vec{k}, \omega\right) \tag{13}$$

(a) Fourier transform the expressions of the fields in terms of the potentials

$$\vec{E}\left(\vec{x}, t\right) = -\vec{\nabla}\Phi\left(\vec{x}, t\right) - \frac{\partial}{\partial t}\vec{A}\left(\vec{x}, t\right) \; , \quad \vec{B}\left(\vec{x}, t\right) = \vec{\nabla} \times \vec{A}\left(\vec{x}, t\right) \tag{14}$$

to obtain the expressions of $\vec{E}\left(\vec{k}, \omega\right)$ and $\vec{B}\left(\vec{k}, \omega\right)$ in terms of $\Phi\left(\vec{k}, \omega\right)$ and $\vec{A}\left(\vec{k}, \omega\right)$. Fourier transform all of Maxwell's equations and use them to obtain *linear* relations between $\Phi\left(\vec{k}, \omega\right)$, $\vec{A}\left(\vec{k}, \omega\right)$, $\rho\left(\vec{k}, \omega\right)$, and $\vec{J}\left(\vec{k}, \omega\right)$.

(b) Solve these linear relations for $\Phi\left(\vec{k}, \omega\right)$ and $\vec{A}\left(\vec{k}, \omega\right)$ in terms of $\rho\left(\vec{k}, \omega\right)$, and $\vec{J}\left(\vec{k}, \omega\right)$, assuming the *Lorentz condition in the medium*: $\vec{k} \cdot \vec{A}\left(\vec{k}, \omega\right) = \mu_0 \; \omega \; \varepsilon(\omega) \; \Phi\left(\vec{k}, \omega\right)$.

Next, consider a point particle of charge q moving through the medium at *constant* velocity $\vec{v}$, so

$$\rho\left(\vec{x}, t\right) = q \; \delta\left(\vec{x} - \vec{v}\; t\right) \; , \quad \vec{J}\left(\vec{x}, t\right) = \vec{v}\rho\left(\vec{x}, t\right) \tag{15}$$

(c) Fourier transform these particular charge and current densities to obtain the corresponding $\rho\left(\vec{k}, \omega\right)$ and $\vec{J}\left(\vec{k}, \omega\right)$. Use your results from (b) to find the corresponding $\Phi\left(\vec{k}, \omega\right)$ and $\vec{A}\left(\vec{k}, \omega\right)$.

(d) Use your results for $\Phi\left(\vec{k}, \omega\right)$ and $\vec{A}\left(\vec{k}, \omega\right)$ in (c) to obtain Fourier integral representations of $\Phi\left(\vec{x}, t\right)$ and $\vec{A}\left(\vec{x}, t\right)$. Evaluate these integrals for the special case that $\varepsilon(\omega)$ is a constant ε, in the situation of "super-luminal" particle motion, where $v^2 > \frac{1}{\mu_0 \varepsilon}$. The particle is moving faster through the medium than would freely propagating light waves, in this situation. (Hint: The Fourier integrands actually have singularities in this special situation. Avoid the singularities and evaluate the integrals by *requiring* that the potentials vanish for $\hat{v} \cdot \vec{x} > vt$.)

(e) Your results for $\Phi\left(\vec{x}, t\right)$ and $\vec{A}\left(\vec{x}, t\right)$ in the special situation in (d) should have singularities along a cone ("shock front") around the straight-line trajectory of the particle. This is known as the Cherenkov radiation cone. Find the opening angle of this cone from your expressions for $\Phi\left(\vec{x}, t\right)$ and $\vec{A}\left(\vec{x}, t\right)$.