$\mathcal{N} = 4$ Yang-Mills scattering amplitudes: from weak to strong coupling

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Z. Bern, L. Dixon, D. Kosower, R. Roiban, M. Spradlin, C. Vergu, AV, arxiv:0803.1465

F. Cachazo, M. Spradlin, AV, arxiv:0805.4832

M. Spradlin, AV, C. Wen, arxiv:0808.1054

Outline

- Motivations.
- Refresher on the calculation of gluon amplitudes.
- Basis of integrals and dual conformal invariance.
- Leading singularity method.
- Two loop six point amplitude.
- Verdict on BDS ansatz and amplitude/Wilson loop relation.
- Conclusions and open questions.

Scattering Amplitudes

● **Gluon scattering amplitudes** in QCD and supersymmetric gauge theories are very difficult to compute, so this is a fertile ground for new insights and methods.

● Experimental program at the **LHC requires many new calculations** of QCD-associated processes. Development of new tools for computing scattering amplitudes is an important topic.

Scattering Amplitudes

• **Gluon scattering amplitudes** in QCD and supersymmetric gauge theories are very difficult to compute, so this is a fertile ground for new insights and methods.

• We have learned that Feynman diagrams are not the most efficient way to calculate scattering amplitudes: too messy+too many terms+hide the structure of amplitudes.

• There has been a lot of progress on tree amplitude calculations. [Bern, Dixon, Kosower] [Witten] [Cachazo, Svrcek, Witten] [Britto, Cachazo, Feng] [Roiban, Spradlin, AV] [Brandhuber, Spence, Travaglini] [Dixon, Glover, Khoze] [Bern, Dixon, Kosower] [Arkani-Hamed, Kaplan] [many others]

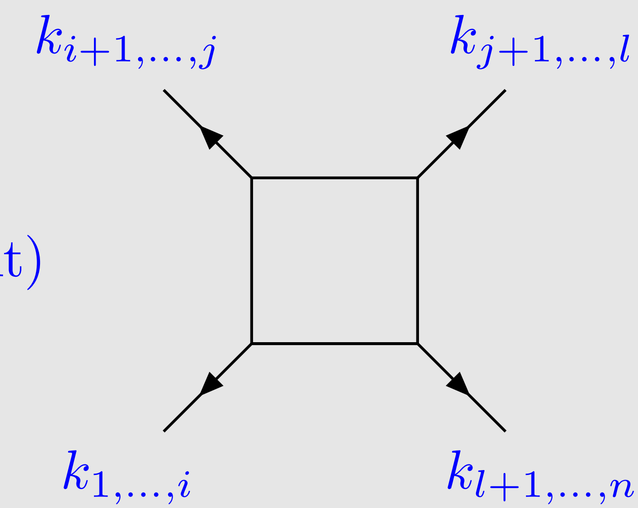
All tree level perturbative amplitides are under control.

One-Loop Amplitudes

In the $\mathcal{N} = 4$ theory, the problem of computing any one-loop amplitude can be reduced to that of computing tree amplitudes. (1990-2004)

Scalar box integrals provide a complete basis for all one-loop gluon amplitudes in $\mathcal{N} = 4$ [Bern, Dixon, Kosower].

$$\mathcal{A}^{1-\text{loop}} = \sum_{\text{boxes}} (\text{coefficient})$$



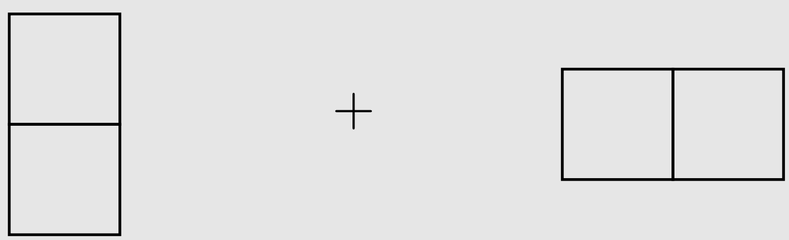
Generalized unitarity methods can be used to determine the coefficients for a desired amplitude [Bern, Dixon, Kosower][Britto, Cachazo, Feng].

Higher Loops

Unitarity based methods for computing the coefficients can be generalized to higher loop amplitudes [Bern, Dixon, Smirnov, 2005] [Buchbinder, Cachazo, 2005] [Bern, Czakon, Dixon, Kosower, Smirnov, 2006] [Bern, Carrasco, Johansson, Kosower, 2007]

Unfortunately a complete basis of integrals is not known even for all two-loop amplitudes...

For example, the two-loop four-particle amplitude is given by the sum of only two scalar integrals [Bern, Rozowsky, Yan, 1997]



But in general it is not trivial to determine which integrals contribute to any particular amplitude.

The Method

In my talk I will describe a method called the leading singularity method [Cachazo, 2008], [Cachazo, Spradlin, AV, 2008], which is a refinement of [Buchbinder, Cachazo, 2005], [Bern, Carrasco, Johansson, Kosower, 2007] [Cachazo, Skinner, 2008]. Via this set of techniques,

- a natural basis of integrals is provided (does not coincide with dual conformally invariant basis),

- the coefficients are determined by solving simple linear equations,

- and these linear equations are easy to write down by hand (for MHV at least).

Basic idea: Feynman diagrams possess singularity which must be reproduced by any representation of the amplitude in terms of simpler integrals.

The Target

Much of what I will say in this talk will be more general, but the specific target of our calculation is the two-loop six-particle MHV amplitude in $\mathcal{N} = 4$ super-Yang Mills theory.

The parity-even part of this amplitude was recently presented in [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, AV, 2008]

The parity-odd part was presented in [Cachazo, Spradlin, AV, 2008]

In fact the helicity information (MHV versus non-MHV) appears only in the homogeneous terms of linear equations, so much of the work done for the MHV amplitude can be applied directly to NMHV [in progress]

Three-loop five-point amplitude [Spradlin, AV, Wen, 2008]

Background: Calculation of Amplitudes

Any L-loop scattering amplitude can, in principle, be obtained by summing over all Feynman diagrams:

$$\mathcal{A}^{(L)}(p) = \int d\ell_1 \cdots d\ell_L \sum_j F_j(p,\ell) \qquad (1)$$

$p =$ **external momenta**

$\ell =$ **loop momenta**

However, in practice this is a hopeless exercise due to the enormously large number of Feynman diagrams and their complexity in Yang-Mills theory.

Background: Calculation of Amplitudes

$$\mathcal{A}^{(L)}(p) = \int d\ell_1 \cdots d\ell_L \sum_j F_j(p,\ell) \tag{2}$$

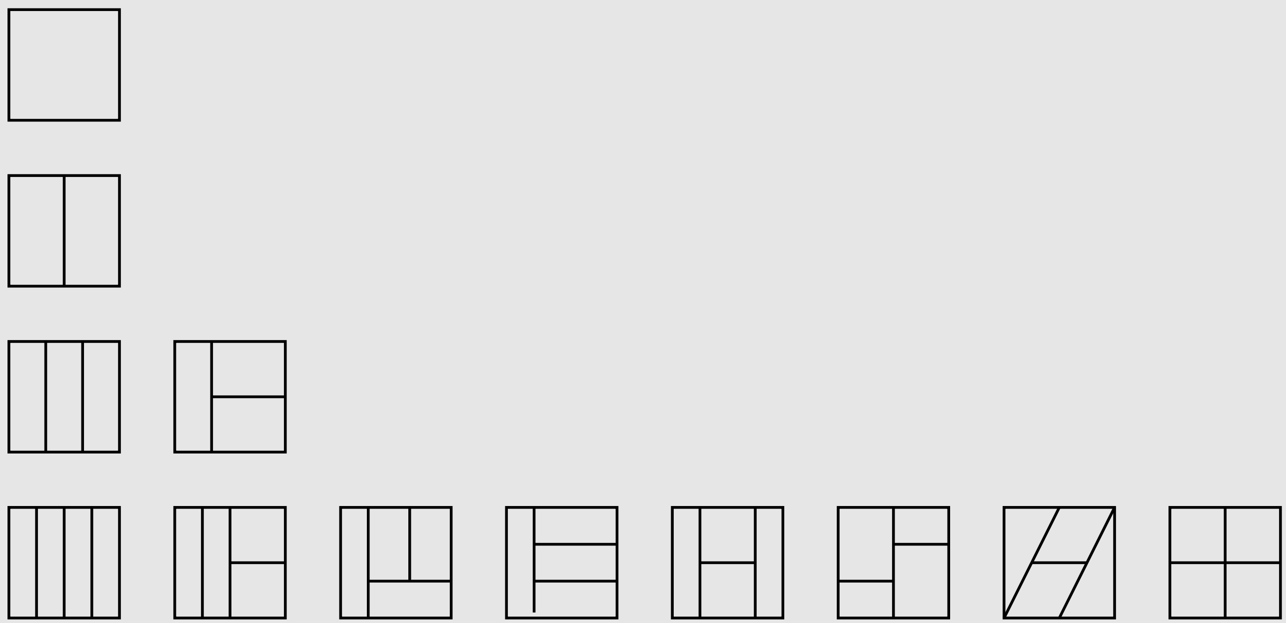
Rather, calculations typically proceed by first finding a representation of the amplitude in terms of a relatively simple basis of integrals $\{I_i\}$:

$$\mathcal{A}^{(L)}(p) = \sum_i c_i(p) \int d\ell_1 \cdots d\ell_L \, I_i(p,\ell) \tag{3}$$

where the coefficients $c_i(p)$ are computed by other means, such as the unitarity-based method [Bern, Dixon, Kosower, 1990s] or maximal cuts [Buchbinder, Cachazo] [Bern, Carrasco, Johansson, Kosower, 2007] .

Example: Four external particles

For example, unitarity based methods were used to express the four-particle amplitude in $\mathcal{N} = 4$ Yang-Mills as the sum of the following scalar integrals:



[Bern, Rozowsky, Yan, 1997] [Bern, Dixon, Smirnov, 2005]

[Bern, Czakon, Dixon, Kosower, Smirnov, 2006]

A Difficulty

One important difficulty is that there is no known basis of integrals in the general case. Only in some special cases is a basis known:

• one-loop, any number of external particles;

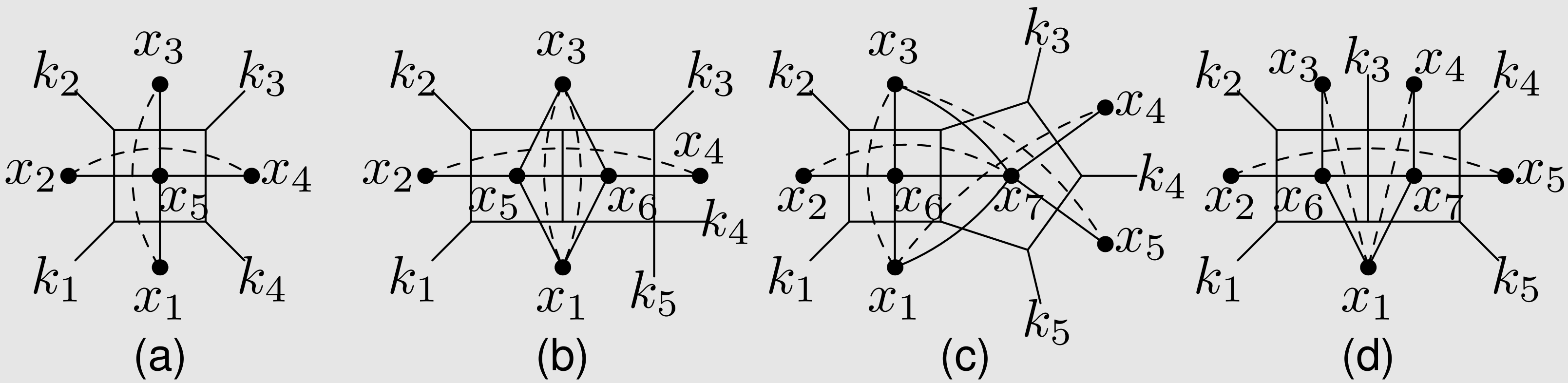
$\Rightarrow$ scalar box integrals, as discussed above...

• and a very plausible conjecture exists for four particles at any number of loops which has emerged from the work of [Bern, Czakon, Dixon, Drummond, Henn, Korchemsky, Kosower, Smirnov, Sokatchev, and others, 2006-2007].

$\Rightarrow$ dual conformal integrals

• higher number of particles? Parity odd parts of two-loop amplitudes are not given in terms of dual conformally invariant integrals. [Bern, Dixon, Kosower, Roiban, Smirnov] [Cachazo, Spradlin, AV]

Examples of Dual Conformal Integrals

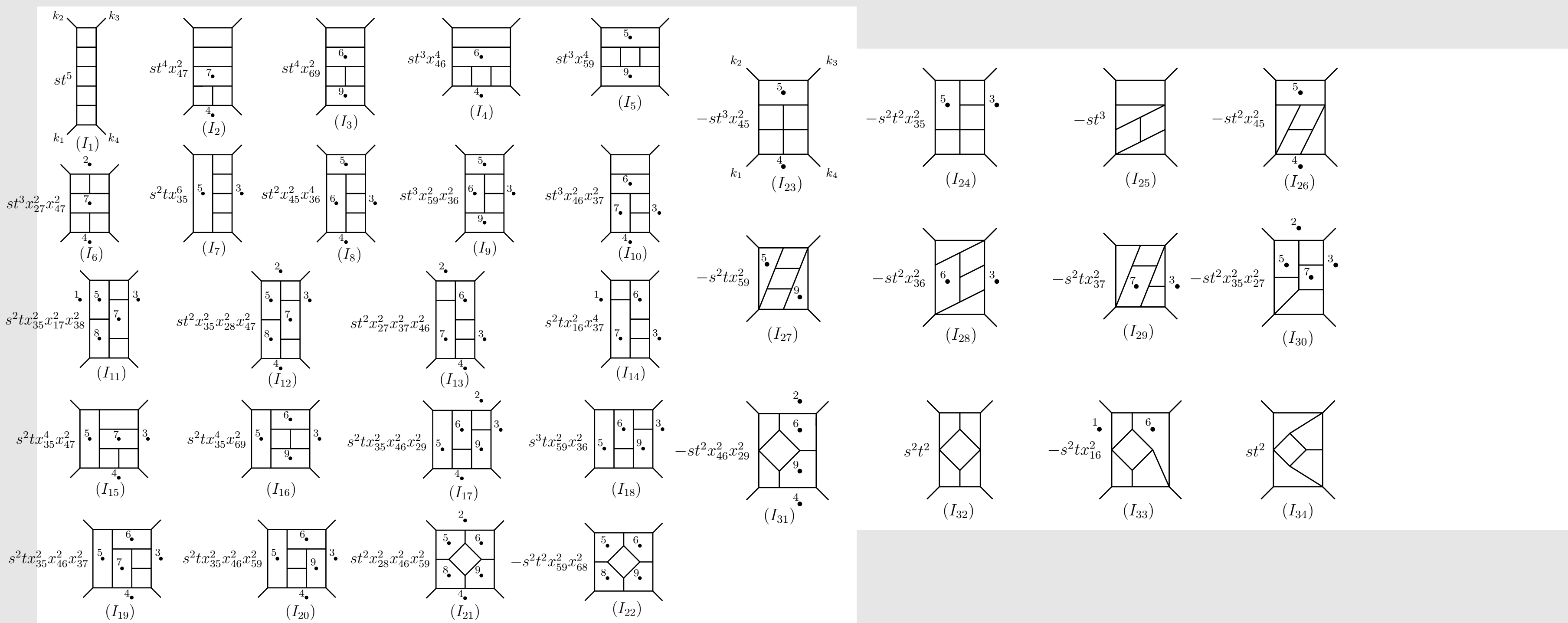


- **Points x_i label the vertices of the dual graph, a solid line connecting two points x_i and x_j corresponds to a factor of $1/x_{ij}^2$, while a dashed line corresponds to a factor of x_{ij}^2.**

- **An integral is dual conformal invariant if the difference between the number of solid lines and dashed lines at a vertex equals 4 at the internal vertices and 0 at the external vertices.**

Dual Conformal Invariant Diagrams at Five Loops

[Bern, Carrasco, Johansson, Kosower, 2007]



Leading Singularity Method

Each individual singularity provides enough linear equations to determine higher-point and multi-particle equations:

each pole gives a different equation!

- **Contours: inhomogenous part of equations**
- **Geometric integrals: homogeneous part of equations**

Determining the Integrand

The idea is to look at the equation

$$\sum_i c_i(p) \int d\ell_1 \cdots d\ell_L \, I_i(p,\ell) = \int d\ell_1 \cdots d\ell_L \sum_j F_j(p,\ell) \tag{4}$$

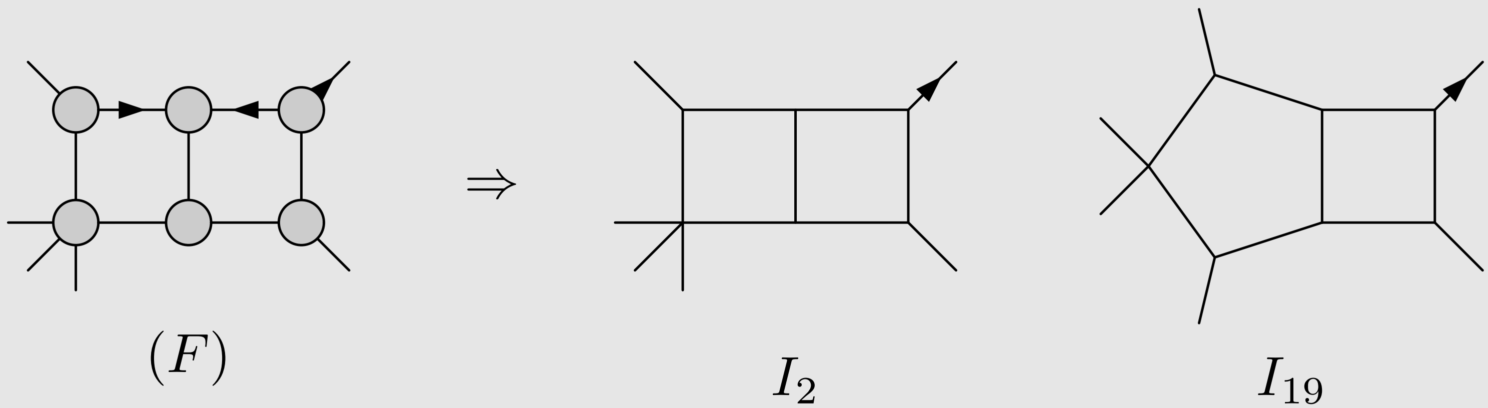
at the level of the integrand, and instead of integrating over the real ℓ-axis in $\mathbb{C}^{4L}$ we integrate over closed contours $\Gamma \subset \mathbb{C}^{4L}$ to obtain linear equations for the desired coefficients!

$$\sum_i c_i(p) \int_\Gamma I_i(p,\ell) = \int_\Gamma \sum_j F_j(p,\ell) \tag{5}$$

We can require this to be true for any contour Γ. By choosing many different contours, we get many different linear equations!

Each contour computes the residue of the integrand on only one of the many isolated singularities.

Example: Topology F



Solving this 2×2 linear system gives the coefficients c_2 and c_{19}. Note that the even and odd-parity parts are determined simultaneously:

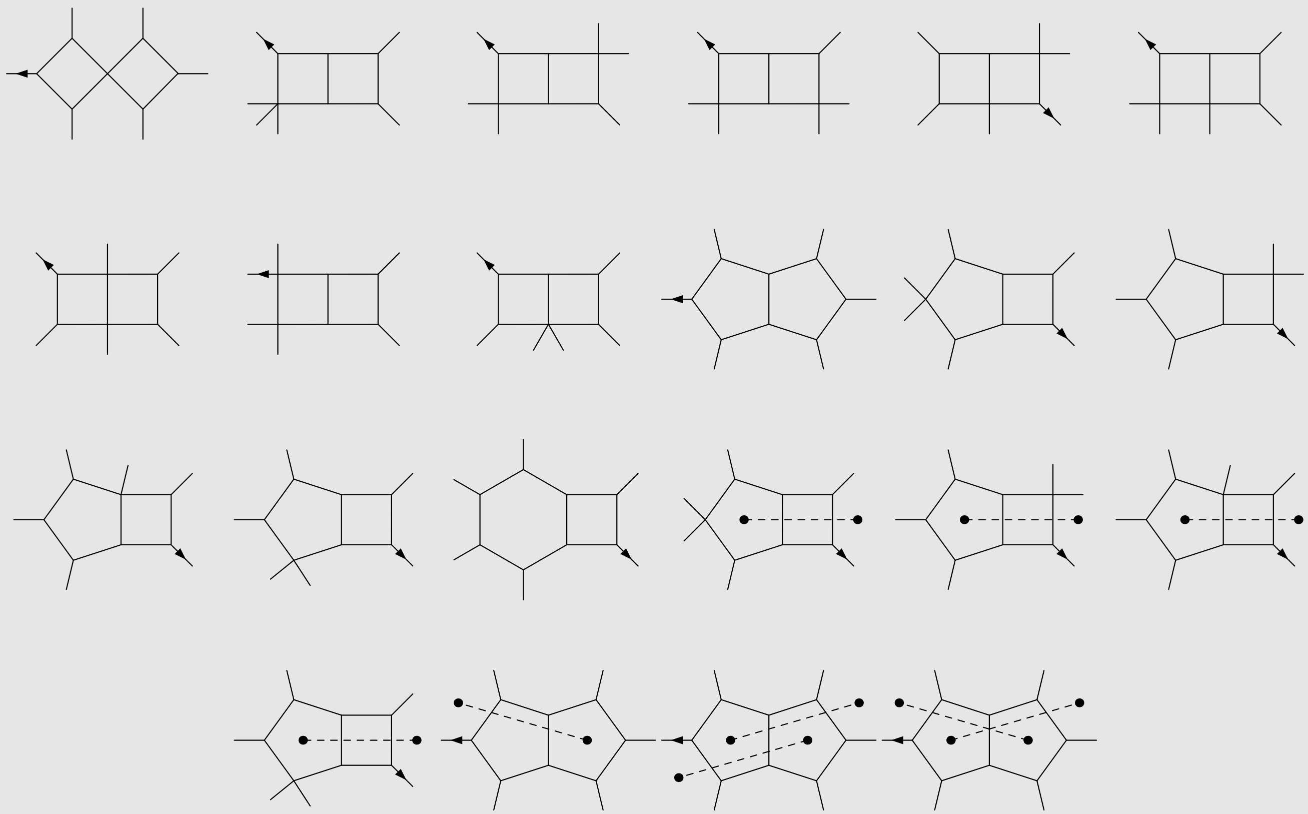
$$\frac{1}{2}(c_2 + \bar{c}_2) = -2 s_{16} s_{12}^2 \,, \qquad \frac{1}{2}(c_2 - \bar{c}_2) = 2 s_{16} s_{12}^2 \left(\frac{a+1}{a-1}\right) \,,$$

$$\frac{1}{2}(c_{19} + \bar{c}_{19}) = 0 \,, \qquad \frac{1}{2}(c_{19} - \bar{c}_{19}) = 4 s_{12}^2 s_{61} \frac{(p^{(1)} + k_{456})^2}{1-a} \,. \qquad (6)$$

where $a = \frac{(p^{(1)} + k_{456})^2}{(p^{(2)} + k_{456})^2}$

Result

We find a representation of the 2-loop six-particle MHV amplitude in terms of



(Several of these can actually be set to zero using reduction identities).

Result

The parity-even part of the amplitude agrees with the recent result of [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, AV, 2008].

The parity-odd part was presented in [Cachazo, Spradlin, AV, 2008]

Note that the full coefficients, both the parity even and parity odd parts, emerge from solving the same linear equations—in in fact it is unnatural to separate the two parts, and we have only done this in order to make the comparison and check our results.

Only the parity even part can be written in a basis with only dual conformal integrals. The leading singularity method naturally provides a set of geometric integrals , which includes also non-dual conformal integrals (these appear in the parity-odd part).

The ABDK/BDS Conjecture

One reason for the recent interest in multi-loop amplitudes in $\mathcal{N} = 4$ super-Yang Mills theory is the ABDK/BDS conjecture, which at two-loops takes the form [Anastasiou, Bern, Dixon, Kosower, 2003]

$$M_n^{(2)}(\epsilon) = \frac{1}{2}(M^{(1)}(\epsilon))^2 - (\zeta(2) + \zeta(3)\epsilon + \zeta(4)\epsilon^2 + \cdots)M^{(1)}(2\epsilon) - \frac{\pi^4}{72} + \mathcal{O}(\epsilon)$$

in dimensional regularization to $D = 4 - 2\epsilon$.

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in dimensional regularization to $D = 4 - 2\epsilon$.

This form is based on explicit computations of two-loop amplitudes for four particles. For five-point amplitude, it has been confirmed by direct calculation [Cachazo, Spradlin, AV, 2006] [Bern, Dixon, Kosower, Roiban, Smirnov, 2006].

BDS Iteration Relations for Multiloop Amplitudes

- This iterative structure together with the exponential nature of IR divergences suggests an all-order resummation should be possible. [Bern, Dixon, Smirnov, 2005]

- Indeed, BDS verified the three-loop generalization for four-particle amplitude by direct calculation, guiding the all-loop order proposal

$$\ln M_n = \sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon)\right)$$

where

$$M_n = \sum_{L=0}^{\infty} a^L M_n^{(L)}(\epsilon),$$

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)},$$

$$a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon}.$$

Why compute two-loop six-point amplitude?

- Alday and Maldacena (2007) have given a prescription for using AdS/CFT to calculate gluon scattering amplitudes at strong coupling.

- Confirmation of the strong coupling prediction from the BDS iteration ansatz for the four-point amplitude and disagreement in the limit of a large number of legs, between the Wilson loop calculation and BDS asnatz.

- Investigations in Regge limit by Tan et all and Lipatov et all (2007)

- Drummond et all and Brandhuber et all (2007) showed that lowest-order contributions to a light-like rectangular Wilson loop agrees with BDS ansatz for gauge theory amplitudes.

- Either the connection between Wilson loops and the amplitudes breaks down? Or BDS ansatz breaks down beyond five-point amplitudes? To answer, one needs six-point Wilson loop and amplitude calculations!

An example of an integral

$$
\begin{aligned}
I^{(12)} \quad = \quad & \frac{(-1)^{1+2\eta} e^{2\epsilon\gamma}}{\Gamma(-1-2\epsilon-\eta)\Gamma(\eta)} \int_{-i\infty}^{+i\infty} \cdots \int_{-i\infty}^{+i\infty} \prod_{j=1}^{18} \frac{dz_j}{2\pi i} \Gamma(-z_j) \\
& \times \frac{\Gamma(3+\epsilon+\eta+z_{1,2,3,4,5,6,7,8,9,10})}{\Gamma(4+\epsilon+\eta+z_{1,2,3,4,5,6,7,8,9,10})} \Gamma(1+z_{3,5,9}) \\
& \times (-s_{12})^{z_{8,13}} (-s_{23})^{z_{14}} (-s_{34})^{z_{1,18}} (-s_{45})^{z_{3,15}} (-s_{61})^{z_{11}} \\
& \times (-s_{123})^{z_{9,16}} (-s_{234})^{z_{17}} (-s_{345})^{z_{2,12}} \\
& \times (-s_{56})^{-5-2\epsilon-2\eta-z_{1,2,3,8,9,11,12,13,14,15,16,17,18}} \\
& \times \frac{\Gamma(-3-\epsilon-z_{1,2,3,4,5,6,7})}{\Gamma(-3-3\epsilon-2\eta-z_{1,2,3,8,9,10})} \\
& \times \frac{\Gamma(5+2\epsilon+2\eta+z_{1,2,3,8,9,10,11,12,13,14,15,16,17,18})}{\Gamma(1-z_4)\Gamma(\eta-z_5)\Gamma(-z_6)\Gamma(1-z_7)} \\
& \times \Gamma(-5-2\epsilon-2\eta-z_{1,2,3,6,8,9,10,11,12,13,14,15,16}) \\
& \times \Gamma(-1-\epsilon-\eta+z_{4,5,6,7}-z_{11,12,14,15,17,18}) \\
& \times \Gamma(-2-\epsilon-\eta-z_{1,2,3,8,9,10}) \Gamma(\eta-z_5+z_{14,15,16}) \\
& \times \Gamma(1-z_4+z_{12,13,18}) \Gamma(1+z_{1,2,4,8}) \\
& \times \Gamma(1-z_7+z_{11,12,15}) \Gamma(1+z_{1,6,10}) \Gamma(1+z_{2,3,7}) \Gamma(1+z_{11,14,17}) \qquad (7)
\end{aligned}
$$

An example of an integral

$$
\begin{aligned}
I^{(12)} = -\frac{1}{\epsilon^4}\Bigg[& \frac{3s_{123}}{(-s_{12})^{1+2\epsilon}\, s_{61}s_{34}s_{45}} + \frac{s_{23}s_{56}}{s_{12}s_{61}s_{34}s_{45}(-s_{234})^{1+2\epsilon}} \\
& + \frac{1}{s_{61}s_{34}(-s_{345})^{1+2\epsilon}}\Bigg] + \frac{1}{\epsilon^3}\Bigg[\frac{s_{123}}{s_{12}s_{61}s_{34}s_{45}} \ln\left(\frac{s_{234}^2 s_{345}^6}{s_{23}s_{34}^3 s_{45}^3 s_{56}}\right) \\
& + \frac{s_{23}s_{56}}{s_{12}s_{61}s_{34}s_{45}s_{234}} \ln\left(\frac{s_{23}s_{56}s_{345}^2}{s_{12}^2 s_{34}s_{45}}\right) + \frac{1}{s_{61}s_{34}s_{345}} \ln\left(\frac{s_{45}s_{234}s_{345}}{s_{23}s_{34}s_{56}}\right) \\
& + \frac{1}{s_{61}s_{34} - s_{234}s_{345}}\, \frac{s_{45}s_{234}s_{12} + 2s_{345}s_{23}s_{56}}{s_{45}s_{234}s_{12}s_{345}} \ln\left(\frac{s_{61}s_{34}}{s_{234}s_{345}}\right) \\
& + \frac{s_{12}s_{45}s_{234} + (s_{23}s_{56} + 3s_{123}s_{234})s_{345}}{s_{12}s_{61}s_{34}s_{45}s_{234}s_{345}} \ln\left(\frac{s_{12}}{s_{61}}\right)\Bigg] + O(\epsilon^{-2})\,,
\end{aligned}
$$

Two-loop six-point amplitude: Results

[Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, AV, 2008]

- **Discrepancy with BDS ansatz.**

- **But agreement with Wilson loop calculations by [Drummond, Henn, Korchemsky and Sokachev, 2008] !!!**

kinematics	(u_1, u_2, u_3)	Δ_A	Δ_W
$K^{(1)}$	$(1/4, 1/4, 1/4)$	-0.0181 ± 0.017	$< 10^{-5}$
$K^{(2)}$	$(0.547253,\ 0.203822,\ 0.88127)$	-2.753 ± 0.012	-2.7553
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-4.74445 ± 0.00653	-4.7446
$K^{(4)}$	$(1/9, 1/9, 1/9)$	4.1161 ± 0.10	4.0914
$K^{(5)}$	$(4/8, 4/81, 4/81)$	9.9963 ± 0.50	9.7255

Two-loop six-point amplitude: Odd part

ABDK/BDS ansatz breaks down for parity-even part of the two-loop six-particle MHV amplitude. [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, AV, 2008]

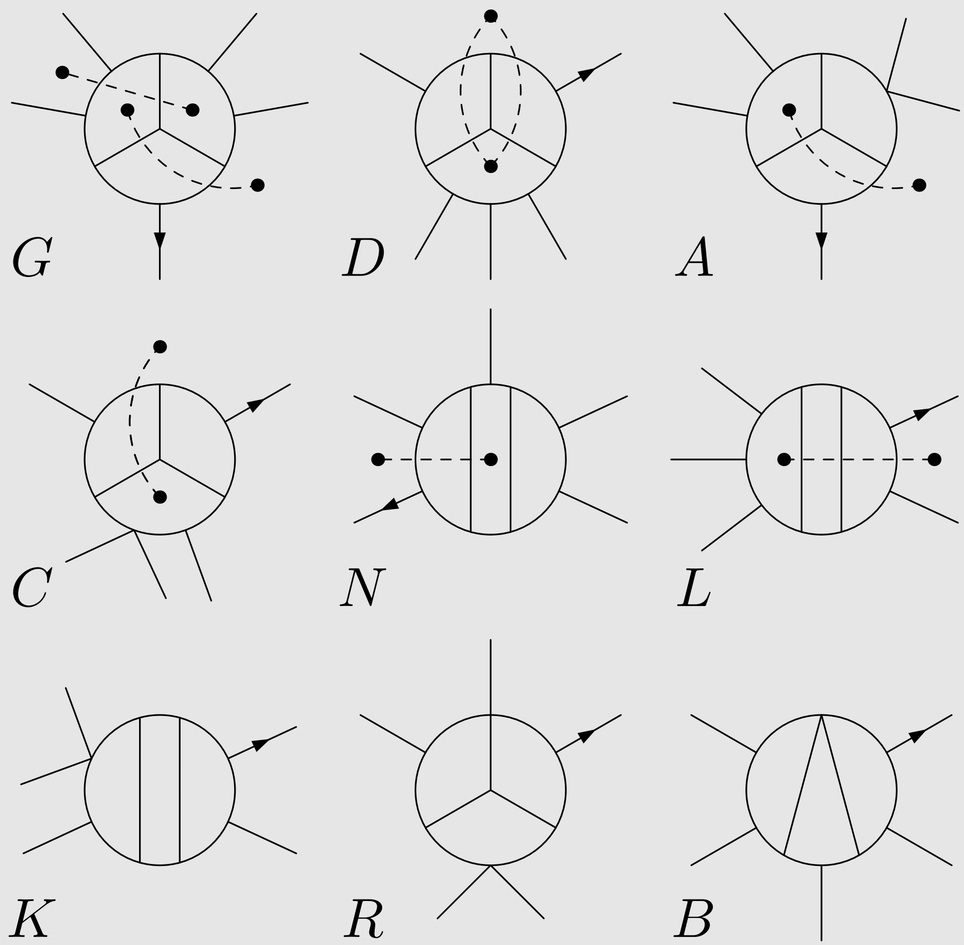
The parity-odd part of the two-loop six-particle amplitude does in fact satisfy ABDK/BDS. [Cachazo, Spradlin, AV, 2008]

For instance, denoting the left- and right-hand sides of ABDK/BDS ansatz by L and R respectively, at randomly generates momenta, we find

$$\begin{aligned} L(\epsilon) &= -\frac{4\times 10^{-16}}{\epsilon^4} + \frac{4\times 10^{-15}}{\epsilon^3} + \frac{1(2)\times 10^{-11}}{\epsilon^2} - \frac{0.430(7)}{\epsilon} - 0.9(1) + \cdots, \\ R(\epsilon) &= -\frac{0.428(2)}{\epsilon} - 0.92(1) + \mathcal{O}(\epsilon) \end{aligned} \tag{8}$$

(It is in fact reasonable to believe that the parity-odd part always satisfies ABDK/BDS, but this remains unproven.)

Three-loop five-point amplitude



The first seven enter the amplitude with coefficient $+1$, the last two with coefficient -1. [Spradlin, AV, Wen, 2008]

Three-loop five-point amplitude

The one, two and three loop "obstructions" are

$$M^{(1)} = -\frac{5}{2}\frac{1}{\epsilon^2} + \frac{5\pi^2}{8} + \frac{179\zeta(3)}{24}\epsilon + \frac{97\pi^4}{1440}\epsilon^2 - \left(\frac{51\pi^2\zeta(3)}{32} - \frac{137\zeta(5)}{8}\right)\epsilon^3 - \cdots$$

$$M^{(2)} = \frac{25}{8}\frac{1}{\epsilon^4} - \frac{35\pi^2}{24}\frac{1}{\epsilon^2} - \frac{865\zeta(3)}{48}\frac{1}{\epsilon} - \frac{97\pi^4}{1152} + \cdots$$

$$M^{(3)} = -\frac{125}{48}\frac{1}{\epsilon^6} + \frac{325\pi^2}{192}\frac{1}{\epsilon^4} + \frac{4175\zeta(3)}{192}\frac{1}{\epsilon^3} + \frac{499\pi^4}{10368}\frac{1}{\epsilon^2} + \cdots$$

These obstructions satisfy the expected BDS relation

$$M^{(3)}(\epsilon) = -\frac{1}{3}(M^{(1)}(\epsilon))^3 + M^{(1)}(\epsilon)M^{(2)}(\epsilon) + f^{(3)}M^{(1)}(3\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)$$

with

$$f^{(3)} = \frac{11\pi^4}{180} + \left(\frac{5\pi^2\zeta(3)}{6} + 6\zeta(5)\right)\epsilon + N_1\epsilon^2, \qquad C^{(3)} = N_2.$$

with $N_1 = 85.263 \pm 0.004$ **and** $N_2 = 17.8241 \pm 0.0009$.

Conclusions

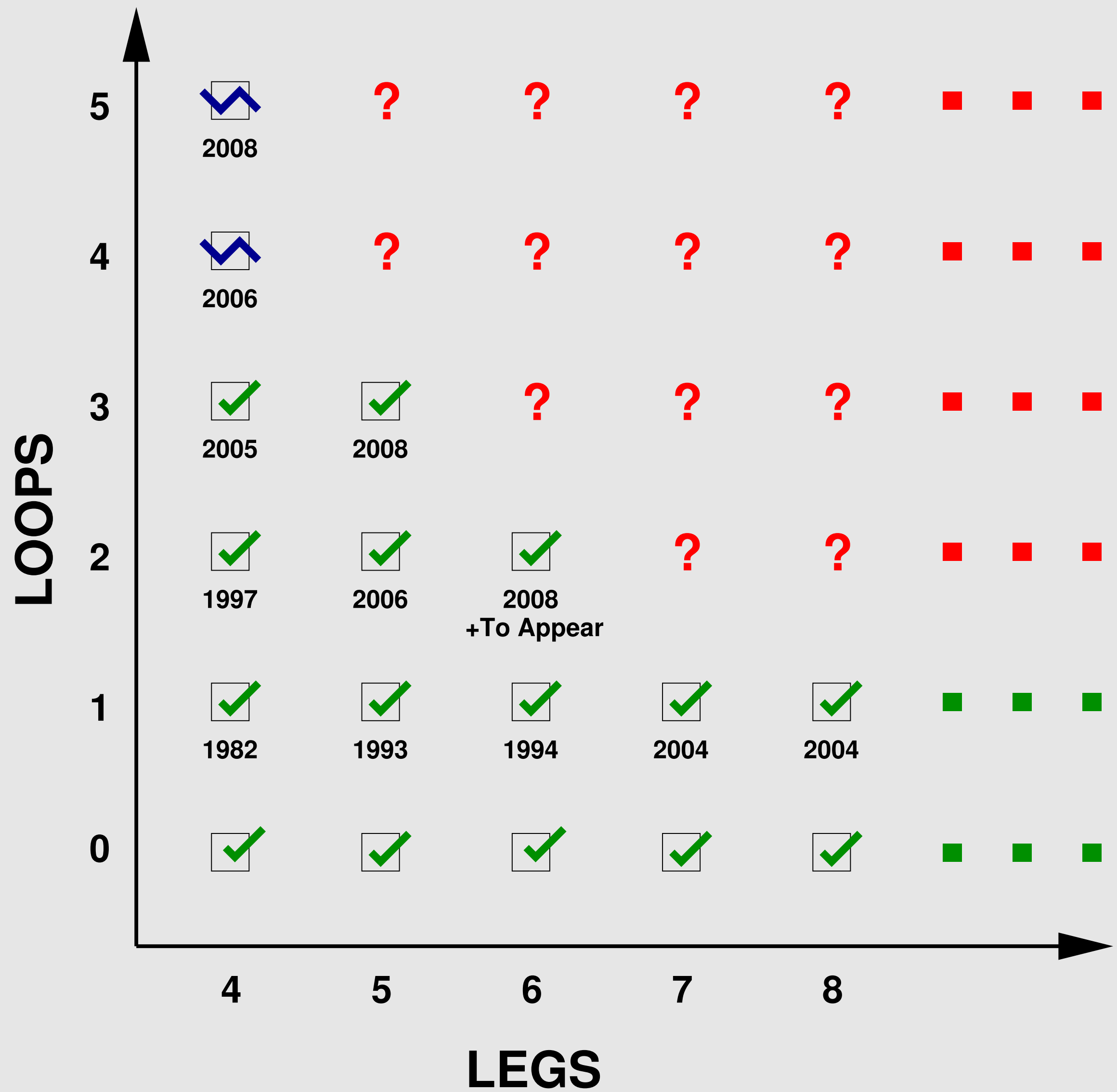
The motivation for our work was two-fold

- To unlock previously hidden mathematical richness lurking deep inside multi-loop gluon amplitudes in $\mathcal{N} = 4$ SYM, and
- To exploit that structure to help simplify otherwise formidable computations.

The leading singularity method provides a relatively simple way to find representations of complicated amplitudes in terms of a simple basis of integrals by just solving linear equations.

One final comment is that the helicity information (MHV versus non-MHV) appears only in the inhomogeneous terms (the right-hand side) of the equations.

$\mathcal{N} = 4$ Yang-Mills Status Report



Open Questions

How far need one calculate before unlocking all the structure?

How much is gained by adding one more loop, or one more leg?

Every new calculation has led to a new surprise!

In the case of loops, there were strong reasons to suspect that special things would start happening at four loops (and they did!) so there was great interest in the calculation of the four-loop cusp anomalous dimension. Five loops: cancellation of $\zeta(6,2)$?

In the case of legs, starting at six-points BDS ansatz breaks down while Wilson loop/amplitude duality holds, suggesting that there should be an additional mechanism besides dual conformal symmetry.

Resummations, non-MHV, non-planar, connection to integrability, other quantities, string theory side, etc.