Dynamical Entanglement Transfer

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I. INTRODUCTION

Consider the Hilbert Space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

A pure state $|\phi\rangle \in \mathcal{H}$ is entangled if it cannot be written as $|\phi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$. Example:

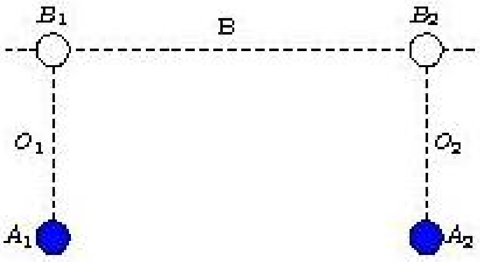
$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|1\rangle + |1\rangle\,|0\rangle)$$

If the state of the total system **AB** is represented by a density matrix ρ. Then ρ is entangled if it cannot be written as

$$\rho = \sum_k p_k \rho_{A,k} \otimes \rho_{B,k} \quad \text{with} \quad p_k > 0 \quad \text{and} \quad \sum_k p_k = 1.$$

Peres Hodorecki criterion: *For 2×2 and 2×3, ρ is entangled if and only if the partial transpose of ρ is negative.*

Consider two systems $\mathbf{A_1}$ and $\mathbf{A_2}$ interacting with a third system $\mathbf{B}$.



We assume that there is <u>no direct</u> coupling between $\mathbf{A_1}$ and $\mathbf{A_2}$.

$$\mathbb{H} = \sum_k g_{1,k} A_{1,k} \otimes B_{1,k} + \sum_k g_{2,k} A_{2,k} \otimes B_{2,k}. \tag{1}$$

Then

$$[B_{1,k}, B_{2,l}] = 0 \Longrightarrow \mathbb{U} = e^{-i\mathbb{H}t} = \mathbb{U}_1 \cdot \mathbb{U}_2$$

In general

$$\rho \longrightarrow \mathbb{U}\rho\mathbb{U}^\dagger$$

We assume that initially systems **A** and **B** are uncorrelated .i.e.

$$\rho = \rho_A \otimes \rho_B.$$

In addition, let the state ρ_A be separable. Tracing out the degrees of freedom of **B** one obtains the reduced density matrix

$$\rho^A = \mathrm{Tr}_B(\mathbb{U}\rho_A \otimes \rho_B\mathbb{U}^\dagger).$$

$\boxed{\rho^A \text{ is in general, a mixed state}}$

$$\rho_B = \sum_k \lambda_{B,k} \left|\phi_{B,k}\right\rangle \left\langle\phi_{B,k}\right|$$

one can express the reduced density matrix as

$$\rho^A = \sum_{i,k} A_{i,k}\rho_A {A_{i,k}}^\dagger$$

$A_{i,j}$ are called Kraus operators and are given by the matrix elements:

$$A_{i,k} = \sqrt{\lambda_{B,k}} \left\langle \Phi_{B,i} \right| \mathbb{U} \left| \phi_{B,k} \right\rangle .$$

These operators satisfy the relation $\sum_{i,k} A_{i,k} {A_{i,k}}^{\dagger} = \mathrm{I}_A$ which guarantees that $\mathrm{Tr}(\rho^A) = 1$. It turns out that if all the operators acting on $\mathcal{H}_B$ commute, i.e.

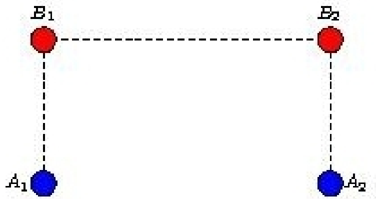
$$[B_{1,k}, B_{1,k'}] = [B_{2,l}, B_{2,l'}] = 0$$

the state ρ^A will remain separable. It is clear that in this case the state ρ^A will be separable because it can be written as a convex sum of products of density operators

$$\rho^A = \sum_{j,k} p_{j,k} \rho_{A_1,jk} \otimes \rho_{A_2,jk} \quad \text{with} \quad \sum_{j,k} p_{j,k} = 1.$$

Some amount of non-commutativity is required!

II. QUBIT-QUBIT INTERACTIONS



Suppose that the "background" **B** system is known to be in the state

$$|\phi_B\rangle = |\psi^-\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle).$$

and **A** is in the separable state $|0,0\rangle$.

$$\mathbb{H}_k = g_k\sigma_{x,k} \otimes \sigma_{x,k}. \quad \text{for} \quad k = 1, 2.$$

Then

$$e^{-ig\sigma_x\otimes\sigma_x} = \begin{pmatrix} \cos(g) & 0 & 0 & -i\sin(g) \\ 0 & \cos(g) & -i\sin(g) & 0 \\ 0 & -i\sin(g) & \cos(g) & 0 \\ -i\sin(g) & 0 & 0 & \cos(g) \end{pmatrix}.$$

$$\rho^A = \begin{pmatrix} \cos^2(g_1)\cos^2(g_2) & 0 & 0 & -\frac{1}{4}\sin(2g_1)\sin(2g_2) \\ 0 & \cos^2(g_1)\sin^2(g_2) & \frac{1}{4}\sin(2g_1)\sin(2g_2) & 0 \\ 0 & \frac{1}{4}\sin(2g_1)\sin(2g_2) & \sin^2(g_1)\cos^2(g_2) & 0 \\ -\frac{1}{4}\sin(2g_1)\sin(2g_2) & 0 & 0 & \sin^2(g_1)\sin^2(g_2) \end{pmatrix}$$

which is separable because it is invariant under partial transposition + $g_i \longrightarrow -g_i$.

HOWEVER

If the state of the composite system $\mathbf{A_1B_1}$ is initially $|0,0\rangle$, then it will evolve under the action of $\mathbb{U}_1$ into

$$|\phi\rangle = \cos(g)\,|0,0\rangle - i\sin(g)\,|1,1\rangle$$

which is entangled for all the values of "g" such that $\cos(g) \neq 0$ or $\sin(g) \neq 0$.

Another Example

$$\mathbb{H}_k = g_k \vec{\sigma}_{A,k} \cdot \vec{\sigma}_{B,k}$$

$$\mathbb{U}_1 = e^{-ig_1 \vec{\sigma}_{A_1} \otimes \vec{\sigma}_{B_1}} = \begin{pmatrix} e^{-ig_1} & 0 & 0 & 0 \\ 0 & e^{ig_1}\cos(2g_1) & -ie^{ig_1}\sin(2g_1) & 0 \\ 0 & -ie^{ig_1}\sin(2g_1) & e^{ig_1}\cos(2g_1) & 0 \\ 0 & 0 & 0 & e^{-ig_1} \end{pmatrix}$$

Then

$$\rho^A = \frac{1}{2} \begin{pmatrix} \cos^2(2g_1) + \cos^2(2g_2) & 0 & 0 & 0 \\ 0 & \sin^2(2g_2) & -e^{-2i(g_1-g_2)}\sin(2g_1)\sin(2g_2) & 0 \\ 0 & -e^{2i(g_1-g_2)}\sin(2g_1)\sin(2g_2) & \sin^2(2g_1) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For matrices of the form

$$\rho^A = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}$$

The negative eigenvalue is given by

$$\lambda_{neg}(g_1 g_2) = \frac{1}{2}(\rho_{11} + \rho_{44} - \sqrt{(\rho_{11} + \rho_{44})^2 + 4(|\rho_{23}|^2 - \rho_{11}\rho_{44})})$$

and it has the following series expansion in g_1 and g_2.

$$\lambda_{neg}(g_1, g_2) = -4g_1^2 g_2^2 - \frac{8}{3}(g_1^4 g_2^2 + g_1^2 g_2^4) + \frac{112}{45}(g_1^6 g_2^2 + g_1^2 g_2^6) - \frac{16}{9} g_1^4 g_2^4 + O(g^{10})$$

Notice that the first contribution to the negativity of the partial transposed density matrix comes from fourth order in the power series expansion.

III. ENTANGLEMENT FROM FOCK SPACE STATES

The reduced density matrix reads

$$\langle a| \rho^A |a'\rangle = \mathrm{Tr}_B(\rho_B \langle \phi_A| \mathbb{U}^\dagger |a'\rangle \langle a| \mathbb{U} |\phi_A\rangle)$$

Assume that system **A** is in the separable state $\phi_A = |0,0\rangle$.

$$\mathbb{K}_i \equiv \langle 0|\mathbb{U}_i|0\rangle \qquad \mathbb{N}_i \equiv \langle 1|\mathbb{U}_i|0\rangle$$

In the basis $\{|a_1\rangle = |0,0\rangle, |a_2\rangle = |0,1\rangle, |a_3\rangle = |1,0\rangle, |a_4\rangle = |1,1\rangle\}$ for $\mathcal{H}_A$ one has the reduced density matrix:

$$\rho^A = \begin{pmatrix} \langle {\mathbb{K}_1}^\dagger \mathbb{K}_1 {\mathbb{K}_2}^\dagger \mathbb{K}_2\rangle & \langle {\mathbb{K}_1}^\dagger \mathbb{K}_1 {\mathbb{N}_2}^\dagger \mathbb{K}_2\rangle & \langle {\mathbb{N}_1}^\dagger \mathbb{K}_1 {\mathbb{K}_2}^\dagger \mathbb{K}_2\rangle & \langle {\mathbb{N}_1}^\dagger \mathbb{K}_1 {\mathbb{N}_2}^\dagger \mathbb{K}_2\rangle \\ \langle {\mathbb{K}_1}^\dagger \mathbb{K}_1 {\mathbb{K}_2}^\dagger \mathbb{N}_2\rangle & \langle {\mathbb{K}_1}^\dagger \mathbb{K}_1 {\mathbb{N}_2}^\dagger \mathbb{N}_2\rangle & \langle {\mathbb{N}_1}^\dagger \mathbb{K}_1 {\mathbb{K}_2}^\dagger \mathbb{N}_2\rangle & \langle {\mathbb{N}_1}^\dagger \mathbb{K}_1 {\mathbb{N}_2}^\dagger \mathbb{N}_2\rangle \\ \langle {\mathbb{K}_1}^\dagger \mathbb{N}_1 {\mathbb{K}_2}^\dagger \mathbb{K}_2\rangle & \langle {\mathbb{K}_1}^\dagger \mathbb{N}_1 {\mathbb{N}_2}^\dagger \mathbb{K}_2\rangle & \langle {\mathbb{N}_1}^\dagger \mathbb{N}_1 {\mathbb{K}_2}^\dagger \mathbb{K}_2\rangle & \langle {\mathbb{N}_1}^\dagger \mathbb{N}_1 {\mathbb{N}_2}^\dagger \mathbb{K}_2\rangle \\ \langle {\mathbb{K}_1}^\dagger \mathbb{N}_1 {\mathbb{K}_2}^\dagger \mathbb{N}_2\rangle & \langle {\mathbb{K}_1}^\dagger \mathbb{N}_1 {\mathbb{N}_2}^\dagger \mathbb{N}_2\rangle & \langle {\mathbb{N}_1}^\dagger \mathbb{N}_1 {\mathbb{K}_2}^\dagger \mathbb{N}_2\rangle & \langle {\mathbb{N}_1}^\dagger \mathbb{N}_1 {\mathbb{N}_2}^\dagger \mathbb{N}_2\rangle \end{pmatrix}.$$

$$\langle \hat{\mathbb{B}} \rangle = \mathrm{Tr}(\rho_B \hat{\mathbb{B}})$$

Let us assume that the Hamiltonian describing the interaction between the qubit and system **B** is of the form $\mathbb{H} = \mathbb{H}_1 + \mathbb{H}_2$ with

$$\mathbb{H}_i = \begin{pmatrix} 0 & \mathbb{F}_i{}^\dagger \\ \mathbb{F}_i & 0 \end{pmatrix}, \quad i = (1, 2).$$

If we assume

$$[\mathbb{F}_1, \mathbb{F}_2] = [\mathbb{F}_1, \mathbb{F}_2{}^\dagger] = [\mathbb{F}_1{}^\dagger, \mathbb{F}_2{}^\dagger] = 0$$

then

$$\mathbb{U} = \mathbb{U}_1\mathbb{U}_2, \qquad \mathbb{U}_i = e^{-i\mathbb{H}_i t}.$$

Now, one can write the operators $\mathbb{K}_i$ and $\mathbb{N}_i$ in terms of $\mathbb{F}_i, \mathbb{F}_i{}^\dagger$.

$$\mathbb{K}_i = \langle 0| \, \mathbb{U}_i \, |0\rangle = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k} (\mathbb{F}_i{}^\dagger \mathbb{F}_i)^k = \cos(\sqrt{\mathbb{F}_i{}^\dagger \mathbb{F}_i t})$$

$$\mathbb{N}_i = \langle 1| \, \mathbb{U}_i \, |0\rangle = (-i) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \mathbb{F}_i (\mathbb{F}_i{}^\dagger \mathbb{F}_i)^k = -i\mathbb{F}_i \frac{\sin(\sqrt{\mathbb{F}_i{}^\dagger \mathbb{F}_i t})}{\sqrt{\mathbb{F}_i{}^\dagger \mathbb{F}_i}}.$$

A. Transformations

Let $|\phi_A\rangle = |1,1\rangle$

$$|1,1\rangle = \sigma_x \otimes \sigma_x \, |0,0\rangle \,.$$

Now, the reduced density matrix

$$\langle a| \, \rho^A \, |a'\rangle = \mathrm{Tr}_B(\rho_B \, \langle a_1| \, \sigma_x^\dagger \mathbb{U}_1^\dagger \sigma_x \otimes \sigma_x^\dagger \mathbb{U}_2^\dagger \sigma_x \, |\bar{a'}\rangle \, \langle \bar{a}| \, \sigma_x^\dagger \mathbb{U}_1 \sigma_x \otimes \sigma_x^\dagger \mathbb{U}_2 \sigma_x \, |a_1\rangle)$$

where $|\bar{a}\rangle = \sigma_x \otimes \sigma_x \, |a\rangle$.

$$\sigma_x^\dagger \mathbb{H}_k \sigma_x = \begin{pmatrix} 0 & \mathbb{F}_k \\ {\mathbb{F}_k}^\dagger & 0 \end{pmatrix} .$$

These simple observations lead us to the conclusion that if initially system **A** was known to be in the state $|a_4\rangle = |1,1\rangle \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ then density matrix is given by

$$\rho^A(|a_4\rangle \,, \mathbb{U}) = \begin{pmatrix} \tilde{\rho}_{44} & 0 & 0 & \tilde{\rho}_{41} \\ 0 & \tilde{\rho}_{33} & \tilde{\rho}_{32} & 0 \\ 0 & \tilde{\rho}_{32}^* & \tilde{\rho}_{22} & 0 \\ \tilde{\rho}_{14} & 0 & 0 & \tilde{\rho}_{11} \end{pmatrix}$$

where the matrix elements $\tilde{\rho}_{ij}$ are determined replacing $\mathbb{F}_i$ by ${\mathbb{F}_i}^\dagger$ and vice versa.

The previous discussion can be summarized as

$$\rho^A(|a_4\rangle, \mathbb{U}(\mathbb{F}_k)) = V\rho^A(|a_1\rangle, \mathbb{U}(\mathbb{F}_k^\dagger))V^\dagger$$

where

$$V = \sigma_x \otimes \sigma_x = |a_1\rangle\langle a_4| + |a_4\rangle\langle a_1| + |a_2\rangle\langle a_3| + |a_3\rangle\langle a_2|$$

is the unitary transformation corresponding to the basis permutation $(1,2,3,4) \rightarrow (4,3,2,1)$. Similarly, one can write

$$\rho^A(|a_2\rangle, \mathbb{U}(\mathbb{F}_1, \mathbb{F}_2)) = V\rho^A(|a_1\rangle, \mathbb{U}(\mathbb{F}_1, \mathbb{F}_2^\dagger))V^\dagger$$

with $V = I \otimes \sigma_x$.

Clearly, the entanglement transferred to system $\mathbf{A_1A_2}$ depends on its initial state

IV. SERIES EXPANSION

$$\mathbb{H}_i = \begin{pmatrix} 0 & {\mathbb{F}_i}^\dagger \\ \mathbb{F}_i & 0 \end{pmatrix}, \quad i = (1,2)$$

Consider operators changing the number of particles in the system, that is:

$$\mathbb{F} = \{a_i, a_i + a_j^\dagger, a_i^2 a_j^\dagger + a_i {a_j^\dagger}^2, \ldots\}$$

then

$$\rho^A = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{pmatrix}.$$

Series expansion for the eigenvalues of $\rho^{T_{A_1}}$

The state ρ^A will be entangled if one of the following expressions is positive

$$n_{23} = t^4(|\langle {\mathbb{F}_1}^\dagger \mathbb{F}_2 \rangle|^2 - \langle {\mathbb{F}_1}^\dagger \mathbb{F}_1 {\mathbb{F}_2}^\dagger \mathbb{F}_2 \rangle)$$
$$n_{14} = t^4(|\langle {\mathbb{F}_1}^\dagger {\mathbb{F}_2}^\dagger \rangle|^2 - \langle {\mathbb{F}_1}^\dagger \mathbb{F}_1 \rangle \langle {\mathbb{F}_2}^\dagger \mathbb{F}_2 \rangle)$$

For the particular case in which the operators $\mathbb{F}_i$ are linear combinations of creation and annihilation operators we have

$$[\mathbb{F}_i, {\mathbb{F}_i}^\dagger] = c_i \quad \text{for} \quad (i = 1, 2),$$

where c_i is a c-number. In this case one can show that n_{23} and n_{14} are negative if either $c_1 < 0$ or $c_2 < 0$. For example if $c_2 < 0$ then making use of Schwarz inequality one has

$$n_{14} = t^4(|\langle {\mathbb{F}_1}^\dagger {\mathbb{F}_2}^\dagger \rangle|^2 - \langle {\mathbb{F}_1}^\dagger \mathbb{F}_1 \rangle \langle \mathbb{F}_2 {\mathbb{F}_2}^\dagger \rangle + c_2 \langle {\mathbb{F}_1}^\dagger \mathbb{F}_1 \rangle) \leq c_2 t^4 \langle {\mathbb{F}_1}^\dagger \mathbb{F}_1 \rangle \leq 0$$

Analogously, for n_{23} we have

$$n_{23} = t^4(|\langle {\mathbb{F}_1}^\dagger \mathbb{F}_2 \rangle|^2 - \langle ({\mathbb{F}_1}^\dagger \mathbb{F}_2)({\mathbb{F}_1}^\dagger \mathbb{F}_2)^\dagger \rangle + c_2 \langle {\mathbb{F}_1}^\dagger \mathbb{F}_1 \rangle) \leq c_2 t^4 \langle {\mathbb{F}_1}^\dagger \mathbb{F}_1 \rangle \leq 0$$

where we used the operator identity $\langle \mathbb{A}\mathbb{A}^\dagger \rangle \geq |\langle \mathbb{A} \rangle|^2$.

V. EXAMPLES

JC Model

$$\mathbb{H}_i = \sum_k g_{i,k}(\sigma_+ a_k + \sigma_- {a_k}^\dagger)$$

with

$$\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|\text{ground}\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\text{excited}\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and represents the excited state of the atom. If we denote by $|k\rangle$ the mode created by $a_k^\dagger$ then introducing the state $|\phi_i\rangle = \sum_k \frac{g_{i,k}}{\sqrt{\sum_k g_{i,k}^2}} |k\rangle$ one can rewrite the Hamiltonian as

$$\mathbb{H}_i = g_i(\sigma_+ a(\phi_i) + \sigma_- a^\dagger(\phi_i)) = g_i \begin{pmatrix} 0 & a^\dagger(\phi_i) \\ a(\phi_i) & 0 \end{pmatrix}$$

with $g_i = \sqrt{\sum_k g_{i,k}^2}$.

Let

$$\mathbb{F}_i = g_i a(\phi_i)$$

$$[\mathbb{F}_1, \mathbb{F}_2] = 0, \quad [\mathbb{F}_i, \mathbb{F}_i{}^\dagger] = g_i^2, \quad [\mathbb{F}_1, \mathbb{F}_2{}^\dagger] = g_1 g_2 \langle \phi_1 | \phi_2 \rangle$$

Assume

$$\langle \phi_1 | \phi_2 \rangle = 0$$

A. System with N bosons

Consider N bosons occupying the same state ϕ_B, i.e.

$$|\Phi_B\rangle = \frac{1}{\sqrt{N!}} (a^\dagger(\phi_B))^N |0\rangle$$

$$\begin{aligned}
\langle \mathbb{F}_1{}^\dagger \mathbb{F}_2 \rangle &= g_1 g_2 \langle a_1^\dagger a_2 \rangle = g_1 g_2 N \langle \phi_B | \phi_1 \rangle \langle \phi_2 | \phi_B \rangle \\
\langle \mathbb{F}_1{}^\dagger \mathbb{F}_1 \mathbb{F}_2{}^\dagger \mathbb{F}_2 \rangle &= g_1^2 g_2^2 \langle a_1^\dagger a_1 a_2^\dagger a_2 \rangle = g_1^2 g_2^2 N(N-1) |\langle \phi_1 | \phi_B \rangle|^2 |\langle \phi_2 | \phi_B \rangle|^2.
\end{aligned}$$

$$
\begin{aligned}
n_{23} &= N g_1^2 g_2^2 t^4 |\langle \phi_1 | \phi_B \rangle|^2 |\langle \phi_2 | \phi_B \rangle|^2 > 0 \\
n_{14} &= -N^2 g_1^2 g_2^2 t^4 |\langle \phi_1 | \phi_B \rangle|^2 |\langle \phi_2 | \phi_B \rangle|^2 < 0.
\end{aligned}
$$

B. Mixed States

Let us now consider the special case in which ρ_B represents a statistical mixture of states of the form $|N\rangle = \frac{1}{\sqrt{N!}}(a^\dagger(\phi_B))^N |0\rangle$, that is:

$$
\rho_B = \sum_{N=0}^{\infty} p_N |N\rangle \langle N| \quad \text{with} \quad \sum_{N=0}^{\infty} p_N = 1.
$$

$$
n_{23} = g_1^2 g_2^2 t^4 |u_1|^2 |u_2|^2 ((\sum_{N=0}^{\infty} p_N N)^2 - \sum_{N=0}^{\infty} p_N N(N-1))
$$

which implies that ρ^A will be nonseparable for probability distributions p_N satisfying

$$
(\sum_{N}^{\infty} p_N N)^2 > \sum_{N}^{\infty} p_N N(N-1).
$$

It is interesting to notice that the above condition implies that the probability must be sub-poissonian!. That is:

$$\bar{N} > \bar{N}^2 - (\bar{N})^2 = \sigma_N^2$$

An example is the binomial distribution

$$p_N = \binom{M}{N} p^N (1-p)^{M-N} \quad (\text{for} \quad N = 0, 1, \ldots M)$$

For a state with binomial distribution one obtains

$$n_{23} = g_1^2 g_2^2 t^4 |u_1|^2 |u_2|^2 M p > 0.$$

where $u_i = \langle \phi_i | \phi_B \rangle$.
If $|\phi_A\rangle = |1, 1\rangle$ then the entanglement of $A_1 A_2$ will be determine by n_{23} with the operators $\mathbb{F}_i$ being replaced by $\mathbb{F}_i{}^\dagger$. As expected, in this case we obtain a separable state.

$$|\, \langle \mathbb{F}_1 \mathbb{F}_2{}^\dagger \rangle \,|^2 - \langle \mathbb{F}_1 \mathbb{F}_1{}^\dagger \mathbb{F}_2 \mathbb{F}_2{}^\dagger \rangle = -g_1^2 g_2^2 N(|u_1|^2 + |u_2|^2 - |u_1|^2 |u_2|^2 + \frac{1}{N}) < 0.$$

C. vacuum

Let $\mathbb{F}_i = g_i(\alpha a(\phi_i) + \beta^* a^\dagger(\psi_i))$ with $\langle\phi_1|\phi_2\rangle = \langle\psi_1|\psi_2\rangle = 0$, then

$$[\mathbb{F}_1, \mathbb{F}_2{}^\dagger] = 0$$

In addition, if we assume that $\langle\phi_1|\psi_2\rangle = \langle\phi_2|\psi_1\rangle$, then $[\mathbb{F}_1, \mathbb{F}_2] = 0$.

$$\begin{aligned} \langle N|\mathbb{F}_1{}^\dagger\mathbb{F}_2{}^\dagger|N\rangle &= g_1 g_2 \alpha^* \beta(\langle\psi_1|\phi_2\rangle + N(\langle\psi_2|\phi_B\rangle\langle\phi_B|\phi_1\rangle + \langle\psi_1|\phi_B\rangle\langle\phi_B|\phi_2\rangle)) \\ \langle N|\mathbb{F}_i{}^\dagger\mathbb{F}_i|N\rangle &= g_i^2(|\beta|^2 + N(|\alpha|^2|\langle\phi_i|\phi_B\rangle|^2 + |\beta|^2|\langle\psi_i|\phi_B\rangle|^2)) \end{aligned}$$

If system $\mathbf{B}$ is in the vacuum state $|\phi_B\rangle = |0\rangle$, then setting $N = 0$ in the above equations one obtains

$$n_{14} = g_1^2 g_2^2 t^4 |\beta|^2 (|\alpha|^2 |\langle\psi_1|\phi_2\rangle|^2 - |\beta|^2)$$

indicating that the system $\mathbf{A_1 A_2}$ is entangled for $0 < |\beta| < |\alpha||\langle\psi_1|\phi_2\rangle|$.

VI. EXACT CALCULATIONS

One Boson and $|\phi_A\rangle = |0,0\rangle$

$$\rho_{11} = 1 - \sin^2(g_1 t)|\langle\phi_1|\phi_B\rangle|^2 - \sin^2(g_2 t)|\langle\phi_2|\phi_B\rangle|^2 \quad \rho_{22} = \sin^2(g_2 t)|\langle\phi_2|\phi_B\rangle|^2$$
$$\rho_{33} = \sin^2(g_1 t)|\langle\phi_1|\phi_B\rangle|^2 \qquad \rho_{44} = 0.$$

The only non vanishing off diagonal element of the matrix is

$$\rho_{23} = \sin(g_1 t)\sin(g_2 t)\langle\phi_B|\phi_1\rangle\langle\phi_2|\phi_B\rangle$$

Always Entangled!

$$\mathcal{N}(\rho^A) = \sqrt{(\rho_{11})^2 + 4|\rho_{23}|^2} - \rho_{11} \leq 1.$$

The maximum value of the negativity $\mathcal{N} = 1$ is achieved when $\sin(g_1 t) = \sin(g_2 t) = 1$ and $|\langle\phi_1|\phi_B\rangle| = |\langle\phi_2|\phi_B\rangle| = \frac{1}{\sqrt{2}}$.

One Boson and $|\phi_A\rangle = |1,1\rangle$

$$\begin{aligned}
\rho_{44} &= \cos^2(g_1 t)\cos^2(g_2 t) + |u_1|^2\cos^2(g_2 t)(\cos^2(\sqrt{2}g_1 t) - \cos^2(g_1 t)) \\
&+ |u_2|^2\cos^2(g_1 t)(\cos^2(\sqrt{2}g_2 t) - \cos^2(g_2 t)) \\
\rho_{11} &= \sin^2(g_1 t)\sin^2(g_2 t) - |u_1|^2\sin^2(g_2 t)(\cos^2(\sqrt{2}g_1 t) - \cos^2(g_1 t)) \\
&- |u_2|^2\sin^2(g_1 t)(\cos^2(\sqrt{2}g_2 t) - \cos^2(g_2 t)) \\
\rho_{32} &= u_1 u_2^* \cos(\sqrt{2}g_2 t)\cos(\sqrt{2}g_1 t)\sin(g_1 t)\sin(g_2 t)
\end{aligned}$$

In the case $g_1 = g_2 = g$ and $|u_1| = |u_2| = \frac{1}{\sqrt{2}}$.

$$\sqrt{2}gt \approx n\pi, \quad gt \approx (2m+1)\frac{\pi}{2} \Longrightarrow (2m+1) \approx n\sqrt{2}$$

$$\rho^A \approx |\psi^+\rangle\langle\psi^+|$$

Notice that!

$$5\sqrt{2} = 7.07, \quad 12\sqrt{2} = 16.97, \quad 29\sqrt{2} = 41.01$$

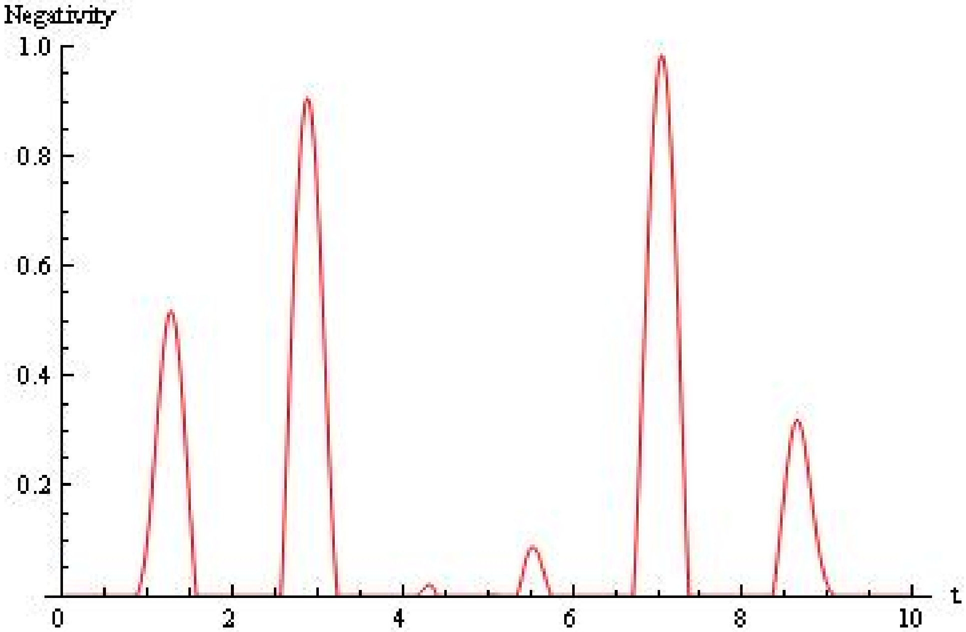


FIG. 1: Negativity for $|\phi_A\rangle = |1,1\rangle$. The maximum value for the negativity in this case is $\mathcal{N} = 0.98$

N bosons

$$\rho_{kk} = \sum_{n,m=0}^{N} \binom{N}{n} \binom{N-n}{m} |u_1|^{2n} |u_2|^{2m} (1 - |u_1|^2 - |u_2|^2)^{N-n-m} F_{kk}(n,m)$$

with

$$\begin{aligned} F_{11}(n,m) &= \cos^2(\sqrt{n} g_1 t) \cos^2(\sqrt{m} g_2 t) \quad & F_{33}(n,m) &= \sin^2(\sqrt{n} g_1 t) \cos^2(\sqrt{m} g_2 t) \\ F_{22}(n,m) &= \cos^2(\sqrt{n} g_1 t) \sin^2(\sqrt{m} g_2 t) \quad & F_{44}(n,m) &= \sin^2(\sqrt{n} g_1 t) \sin^2(\sqrt{m} g_2 t) \end{aligned}$$

$$\rho_{23} = u_1^* u_2 \sum_{n,m=0}^{N-1} N \binom{N-1}{n} \binom{N-1-n}{m} |u_1|^{2n} |u_2|^{2m} (1 - |u_1|^2 - |u_2|^2)^{N-n-m-1} F_{23}(n,m)$$

$$F_{23}(n,m) = \cos(\sqrt{n} g_1 t) \frac{\sin(\sqrt{n+1} g_1 t)}{\sqrt{n+1}} \cos(\sqrt{m} g_2 t) \frac{\sin(\sqrt{m+1} g_2 t)}{\sqrt{m+1}}$$

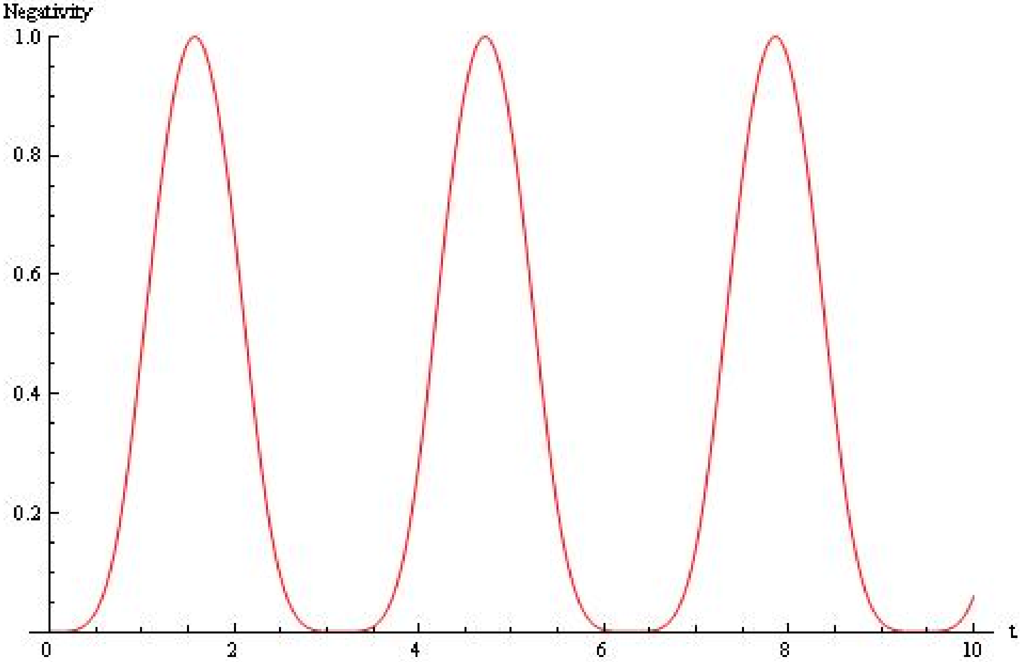


FIG. 2: Negativity vs. Time for N=1

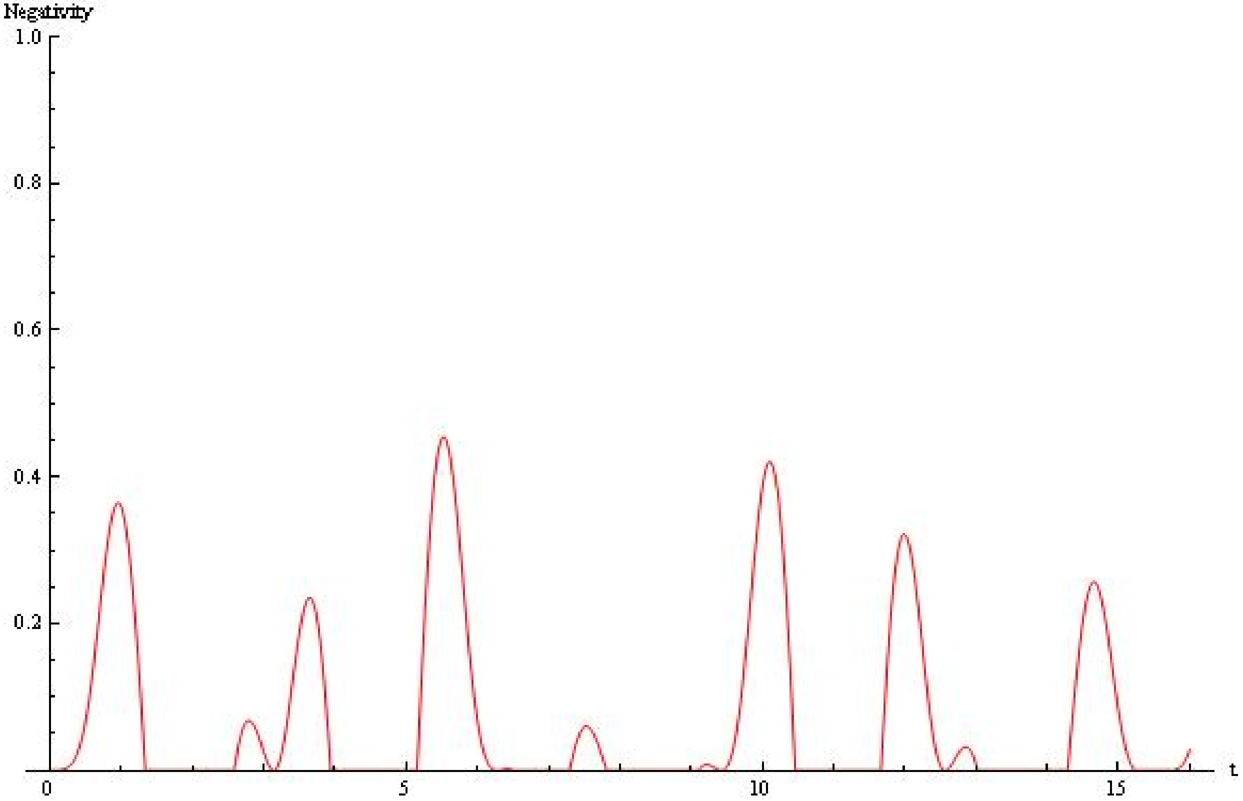


FIG. 3: Negativity vs. Time for N=2

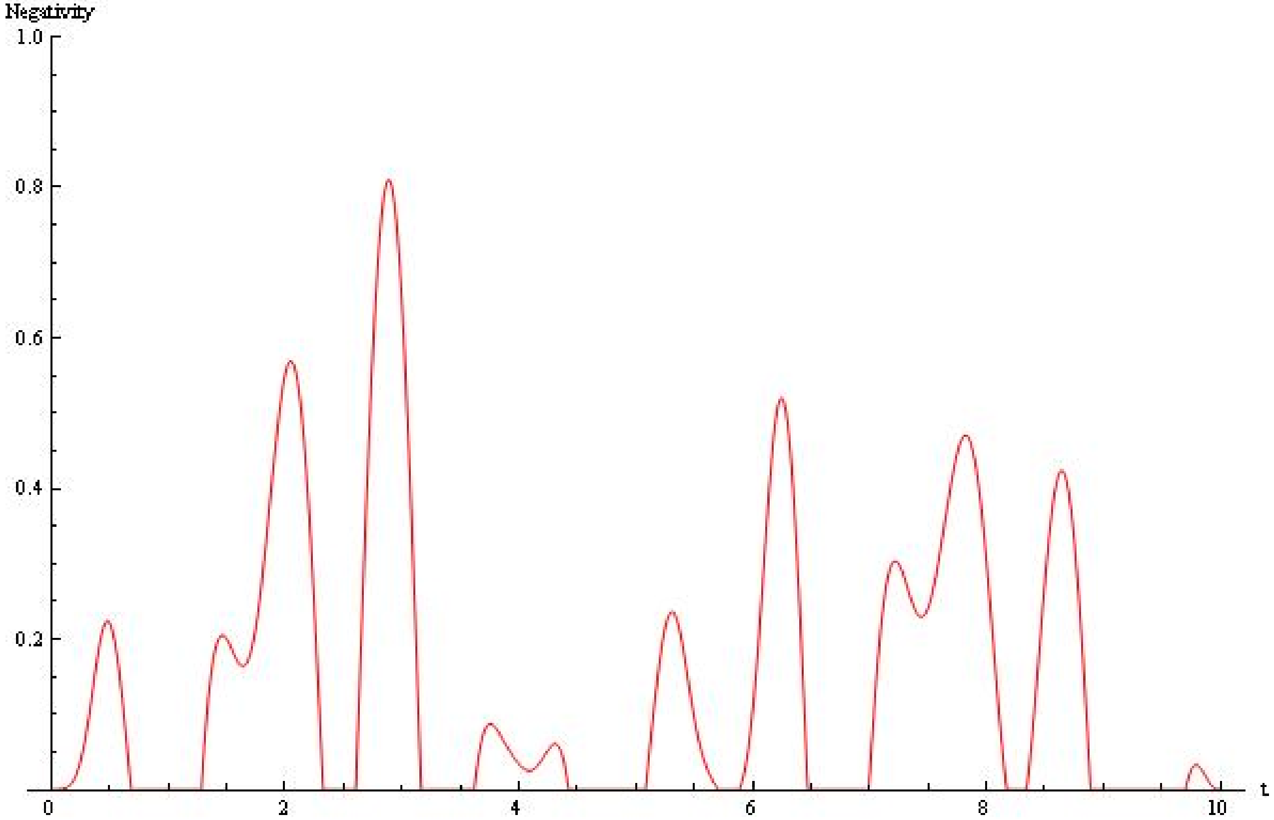


FIG. 4: Negativity vs. Time for N=3

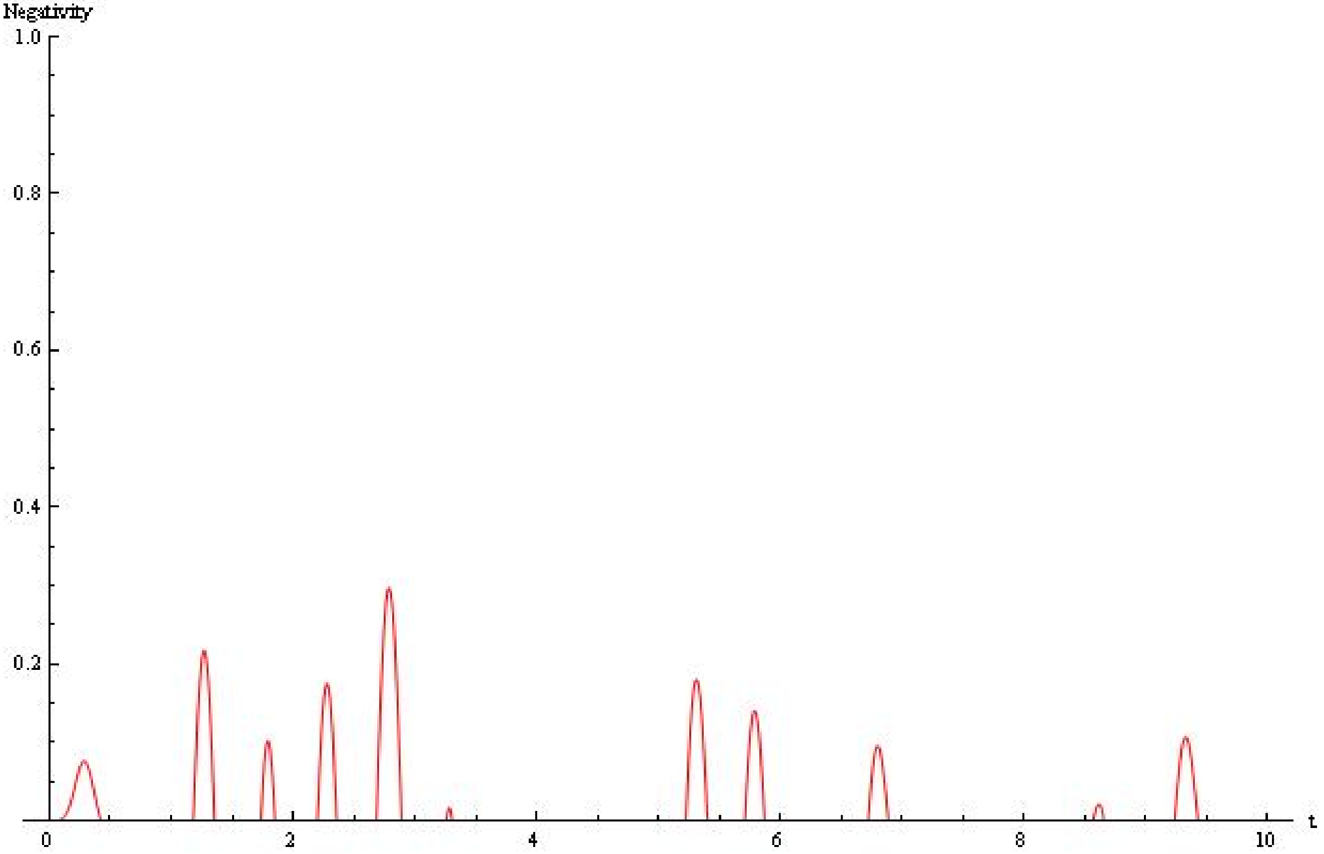


FIG. 5: Negativity vs. Time for N=8

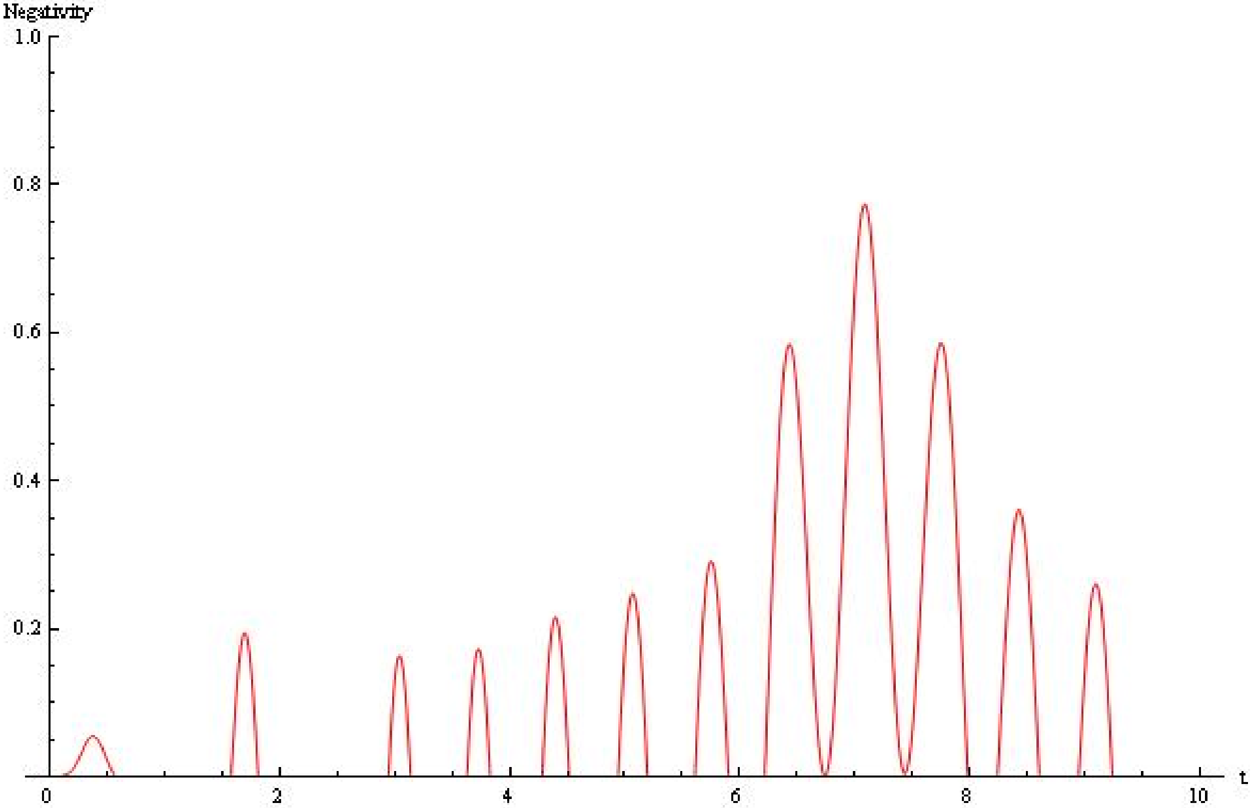


FIG. 6: Negativity vs. Time for N=11

VII. WHERE DOES THE ENTANGLEMENT COME FROM?

Entanglement $\Longrightarrow$ quantum correlation between two subsystems

$$|\Phi_B\rangle = a_B^\dagger\,|0\rangle = u_1\,|1,0,0\rangle + u_2\,|0,1,0\rangle + u_T\,|0,0,1\rangle$$

In the occupation number representation, the state of 1 particle looks like an entangled state