The Tree-Level S-Matrix of Gravity

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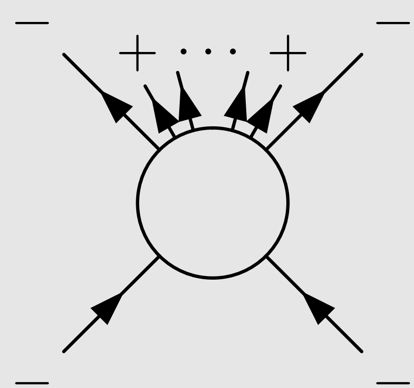
- My talk today will be about tree-level graviton amplitudes. Compared to the enormous progress that has recently been made in unlocking hidden structure of gluon amplitudes, progress on gravity has been much slower...

MHV Amplitudes

In both Yang-Mills and in gravity, ‘maximally helicity violating’ amplitudes are the simplest nonvanishing ones.



This process describes the scattering of two $-$ helicity particles off of a coherent self-dual background field strength; and both self-dual Yang-Mills and self-dual gravity are integrable.

Amplitudes with more opposite helicity particles (in this case $+$) in the outgoing state become more and more complicated.

The Parke-Taylor Formula

MHV gluon scattering is given by the formula

$$A(1,\ldots,2) = \frac{1}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

conjectured by [Parke, Taylor (1986)] and proven by [Berends, Giele (1988)].

Spinor helicity notation:

$$p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}, \qquad \langle i\,j\rangle = \epsilon_{ab}\lambda_i^a \lambda_i^b, \qquad [i\,j] = \epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}_i^{\dot{a}}\widetilde{\lambda}_j^{\dot{b}}.$$

$$(p_i + p_j)^2 = \langle i\,j\rangle[i\,j], \qquad [i|p_j + p_k + \cdots|p_l\rangle = [i\,j]\langle j\,l\rangle + [i\,k]\langle k\,l\rangle + \cdots$$

In contrast, MHV graviton scattering is given by ...

The BGK Formula

In 1987 Berends, Giele and Kuijf conjectured the n-particle MHV amplitude

$$\mathcal{M}_n = \sum_{\mathcal{P}(2,\ldots,n-2)} \frac{[1\,2][n-2\,n-1]}{\langle 1\,n-1\rangle} \left(\prod_{i=1}^{n-3} \prod_{j=i+2}^{n-1} \langle i\,j\rangle \right) \prod_{k=3}^{n-3} [k|p_{k+1}+\cdots+p_{n-1}|n\rangle$$

- This formula has $(n-3)!$ terms
- Secretly it is fully symmetric under the exchange of any two particles, but only an S_{n-3} subgroup of this S_n symmetry is manifest

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Some other variants on this formula have appeared in the literature more recently.

The complexity of these formulas is such that in general it is not possible to prove directly that two different formulas are in fact equivalent — one has to resort to numerical experimentation to convince onself that they are.

The original 1987 BGK formula itself was only proven to be correct in 2008 by Mason and Skinner.

A Comment on Supersymmetry

At tree level, gluon (graviton) amplitudes in QCD (pure Einstein-Hilbert gravity) are identical to those in maximally supersymmetric $\mathcal{N}=4$ super-Yang-Mills ($\mathcal{N}=8$ supergravity).

Nevertheless it is an extremely useful bookkepping device to allow the amplitudes to be expressed in superspace.

At the end of the day, once everything has been calculated, one can choose to look the other way and ignore this 'super' structure, but there is no reason to shoot ourselves in the foot before even taking the first step.

A High-z Limit

In four dimensions, amplitudes are functions of spinor helicity variables

$$(\lambda_i, \widetilde{\lambda}_i)$$

Some gravity amplitudes have exceptionally soft behavior, falling off as $1/z^2$ as two particles are taken to infinity in a particular complex direction

$$\lambda_1 \to \lambda_1 + z\lambda_2, \qquad \widetilde{\lambda}_2 \to \widetilde{\lambda}_2 - z\widetilde{\lambda}_1, \qquad z \to \infty$$

This is a property of amplitudes as a whole — individual Feynman diagrams diverge in this limit, but there is ‘magical’ cancellation.

[Britto, Cachazo, Feng, Witten (2005)], [Arkani-Hamed, Kaplan (2008)]

A High-z Superlimit

The first significant technical advantage of working in superspace with

$$(\lambda_i, \widetilde{\lambda}_i, \eta_i)$$

is that with an appropriately supersymmetrized shift,

$$\eta_1 \to \eta_1 + z\eta_2$$

all graviton amplitudes vanish as $1/z^2$

[Arkani-Hamed, Cachazo, Kaplan (2008)]

(Gluon amplitudes vanish only as $1/z$ in the same limit.)

$1/z$: On-Shell Recursion

$1/z$ vanishing implies on-shell recursion relation

The idea is to consider the contour integral

$$0 = \oint dz \frac{A(z)}{z}$$

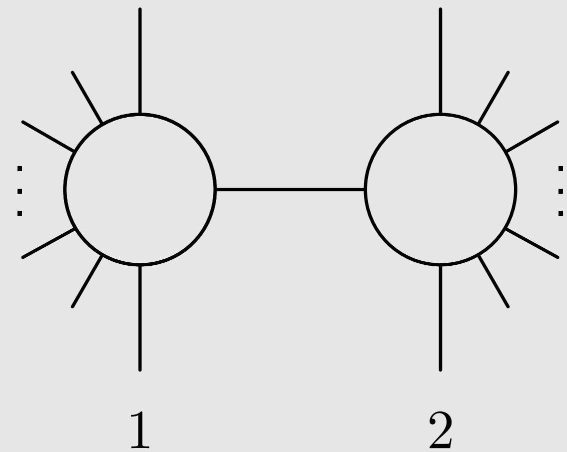
with a contour circling infinity.

Then deform the contour to write it as a sum over residues,

$$0 = A(0) + \sum_k \frac{1}{z_k} \mathrm{Res}_{z=z_k} A(z)$$

Now, the only poles in a tree-level amplitude are those which come from an internal propagator going on-shell. Moreover, the residue at any such pole is a product of two tree amplitudes. Therefore, this formula tells us that

$$A = \sum$$



Any amplitude may be expressed as a sum over all on-shell factorizations in which the two ‘special’ legs 1 and 2 are separated from each other.

Note that this remarkable on-shell recursion only works for theories with spin!

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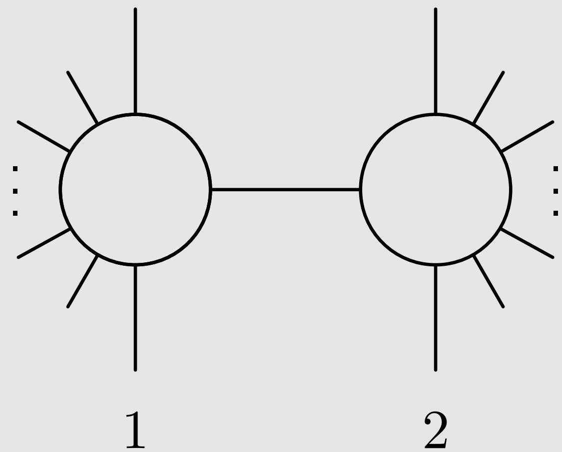

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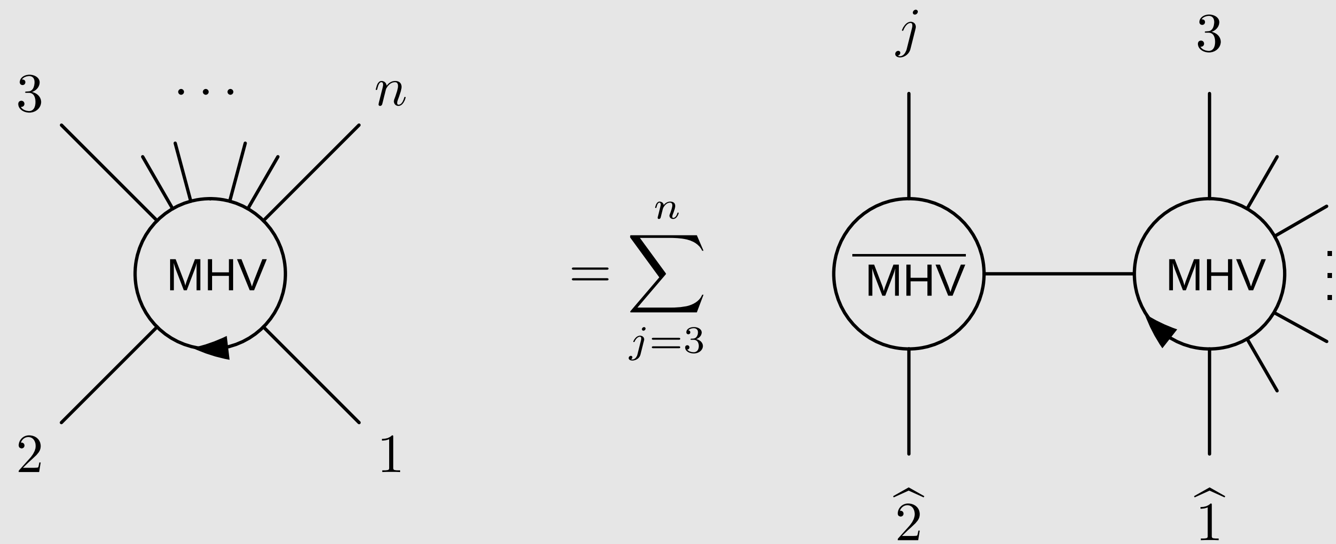
- Scalar field theories have simple Lagrangians but messy amplitudes.
- Gauge & gravity theories have horrendous Lagrangians but very nice amplitudes.

Solving the Recursion for MHV Amplitudes

For MHV amplitudes only a single type of factorization appears, which means that the recursion takes an extremely simple form



which can be solved explicitly. Depending on how one chooses to organize the calculation one can put the result into slightly different forms, [Bedford, Brandhuber, Spence, and Travaglini (2005); Elvang and Freedman (2007)].

Elvang & Freedman

Of particular importance to us will be the formula of Elvang & Freedman,

$$M_n^{\mathrm{MHV}} = \sum_{\mathcal{P}(3,\ldots,n)} [A^{\mathrm{MHV}}(1,2,\ldots,n)]^2 G^{\mathrm{MHV}}(1,2,\ldots,n)$$

where $A(1,2,\ldots,n)$ is nothing but the color-ordered gluon scattering amplitudes in super-Yang-Mills, and $G(1,2,\ldots,n)$ is a particular 'gravity dressing factor.'

The sum is taken over the $(n-2)!$ permutations of the external lines which preserve the two 'special' legs.

It is a generic feature of MHV formulas derived from the on-shell recursion that they have $(n-2)!$ terms.

In contrast, recall that the BGK formula had $(n-3)!$ terms.

$1/z^2$: **Bonus Relations**

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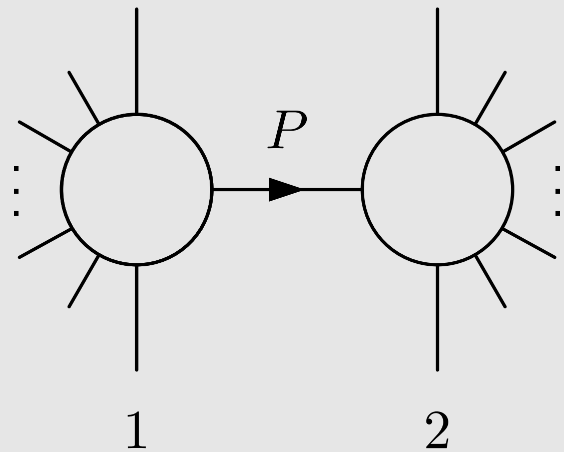
$$0 = \oint dz\, M(z)$$

again with a contour circling infinity.

Deforming the contour to write it as a sum over residues gives the relation

$$0 = \sum_k \mathrm{Res}_{z=z_k} M(z)$$

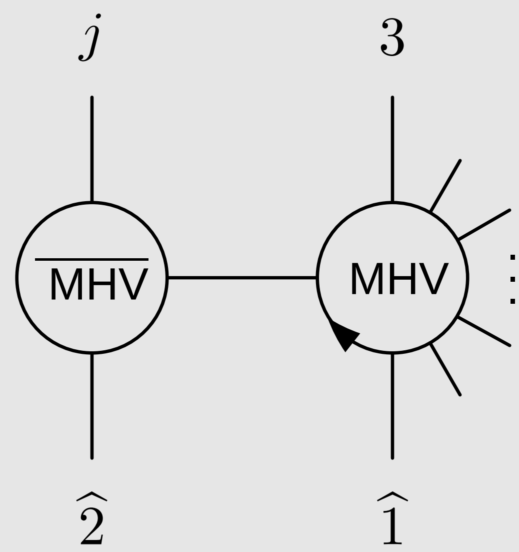
$$0 = \sum \frac{P^2}{[1|P|2\rangle}$$



Cashing in the Bonus Relation for MHV

For MHV amplitudes we have simply

$$0 = \sum_{j=3}^{n} \frac{\langle 1\, j \rangle}{\langle 2\, j \rangle}$$



This linear relation can be used to eliminate (any) one of the $n-2$ terms in the recursion.

When applied recursively, this can be used to reduce $(n-2)!$-term formulas to $(n-3)!$-term formulas...

Summary for MHV

All known formulas in the literature are of two types: they have either $(n-2)!$ or $(n-3)!$ terms.

Until recently, all of the former had been proven by using the on-shell recursion, while all of the latter (like the original BGK formula) were just conjectures (like BGK) that could only be numerically compared to the former.

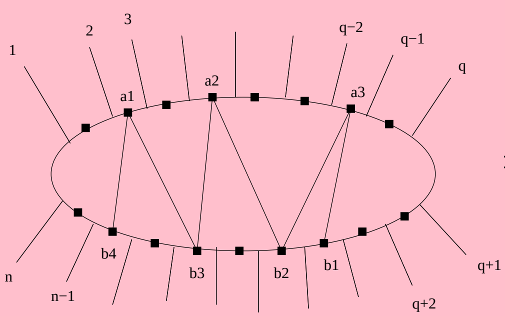
Using the ‘bonus relation’ for gravity amplitudes, we demonstrated the relation between these two classes of formulas and provided the first direct proof that the 1987 BGK formula satisfies the on-shell recursion.

(In August 2008, Mason and Skinner provided a proof of BGK using completely different methods.)

Background: Split Helicity Gluon Amplitudes

In 2005, the nonsupersymmetric on-shell recursion for gluons was solved for the special case of split helicity amplitudes, which form a closed subset under recursion. [Britto, Feng, Roiban, Volovich, MS].

The amplitude $A(1^-, \cdots, p^-, (q+1)^+, \cdots, n^+)$ is given by a sum over all 'zigzag diagrams', of an expression which is trivial to write down:



$$= \frac{\langle q|P_{q,b_1}P_{b_1+1,a_1}P_{a_1+1,b_2}\cdots P_{b_4+1,1}|1\rangle^3}{P^2_{q,b_1}P^2_{b_1+1,a_1}P^2_{a_1+1,b_2}\cdots P^2_{b_4+1,1}[q-1|P_{q,b_1}|b_1\rangle\langle b_1+1|P_{b_1+1,a_1}|a_1]}$$

$$\times \frac{\langle b_1\, b_1+1\rangle\cdots\langle b_4\, b_4+1\rangle[a_1\, a_1+1]\cdots[a_3\, a_3+1]}{\langle q\, q+1\rangle\cdots\langle n\, 1\rangle[2\, 3]\cdots[q-2\, q-1]}$$

Background: Split Helicity Gluon Amplitudes

In 2008 the supersymmetric on-shell recursion for gluons was solved explicitly for all gluon amplitudes in terms of certain ‘dual conformal superinvariants’ [Drummond and Henn].

For example, all next-to-MHV gluon amplitudes are given by

$$A^{\mathrm{NMHV}} = A^{\mathrm{MHV}} \sum_{i=2}^{n-3} \sum_{j=i+2}^{n-1} R_{n;ij}$$

where

$$R_{n;ij} = \frac{\langle i\, i-1\rangle\langle j\, j-1\rangle\, \delta^4(\langle n|x_{ns}x_{st}|\theta_{tn}\rangle + \langle n|x_{nt}x_{ts}|\theta_{sn}\rangle)}{x_{ij}^2\langle n|x_{ni}x_{ij}|j\rangle\langle n|x_{ni}x_{ij}|j-1\rangle\langle n|x_{nj}x_{ji}|i\rangle\langle n|x_{nj}x_{ji}|i-1\rangle}$$

in terms of

$$x_{ij} = p_i + p_{i+1} + \cdots + p_{j-1}, \qquad \theta_{ij} = \lambda_i\eta_i + \cdots + \lambda_{j-1}\eta_{j-1}$$

All Tree-Level Graviton Amplitudes

More complicated gluon amplitudes involve more complicated but still explicitly known R's but the basic idea is the same. Schematically:

$$A^{\rm all} = A^{\rm MHV} \sum_\alpha R_\alpha(\lambda_i, \widetilde{\lambda}_i, \eta_i)$$

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Now recall the Elvang-Freedman formula for MHV graviton amplitudes:

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Inspired by these two results, we were able to solve the on-shell recusion for all gravity amplitudes, obtaining a formula of the form

$$M^{\mathrm{all}} = \sum_{\mathcal{P}(3,\ldots,n)} [A^{\mathrm{MHV}}(1,2,\ldots,n)]^2 \sum_{\alpha} [R_{\alpha}(\lambda_i, \widetilde{\lambda}_i, \eta_i)]^2 G(\lambda_i, \widetilde{\lambda}_i)$$

The listener will be spared the complicated (though explicit) formulas for G.

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Things to note:

• The R's are precisely the same ones that appear in gluon scattering, except they are now squared — this can be understood just because we have moved up from $\mathcal{N}=4$ to $\mathcal{N}=8$ superspace,

$$\delta^4(\theta) \to [\delta^4(\theta)]^2 = \delta^8(\theta)$$

• Other than this minor modification, all of the super-structure of graviton scattering amplitudes is identical to that of gluon amplitudes — the 'gravity dressing factors' G depend only on the **momenta** $(\lambda_i, \widetilde{\lambda}_i)$ and not on the **supermomenta** η_i.

Conclusion

The fast few years have seen enormous procress in our understanding of gluon scattering amplitudes.

This progress has been a pleasing mix of theoretical insights, shedding light on the mathematical structure of amplitudes and their role in AdS/CFT, and more practical results, including impressive new technology for carrying out previously impossible calculations at tree level and beyond.

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In contrast, progress on gravity has been much slower, despite some indications that the theory should be even simpler!

Our results are a small step on a decades-long march towards a better understanding of the structure of graviton scattering, but I am sure that the final chapter is not close to being written.