Aspects of TM(SU)GRA

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- Why 3D? Which model?
- Aspects of 3D chiral gravity
- General supersymmetric solution of topologically massive supergravity
- Attempt to establish the positivity of the energy

Why 3D? Which (SU)GRA model?

- Highly challenging problems in $4D$ quantum gravity, forbiddingly difficult to solve at this time.

  Look for lower dimensional GR where important features of $4D$ gravity exist in a much simpler setting.

- $2D$: There are BHs but no propagating graviton. Thus we need to look at $3D$.

- $3D$: $\int \sqrt{-g}R$: No BH, no propagating graviton

- $3D$: $\int \sqrt{-g}(R - 2\Lambda)$: Has BH ($\Lambda < 0$), but no propagating graviton

- 3D: $\int\sqrt{-g}R + I_{CS}$: Has propagating graviton but no BH.

  $I_{CS} = \int \mathrm{tr}(R \wedge \omega + \frac{2}{3}\omega \wedge \omega \wedge \omega)$

  $\omega = \omega(e)$ via the vanishing torsion.

- 3D: $\boxed{\int\sqrt{-g}(R - 2\Lambda) + I_{CS}}$

  Has BH and propagating graviton!

  In the absence of matter fields, all higher derivative terms can be removed by field redefinition to yield this action (Witten 1988 & 2007)

- If TMG dual to a boundary CFT, can it be solved exactly such that no string theory completion is necessary? Or will string theory be needed one way or another?

Linearized excitations around AdS_3, with standard Brown-Henneaux boundary conditions:

- Central charge at the boundary CFT $(\Lambda = -1/\ell^2)$:

$$c_L = \frac{3}{2G}\left(1 - \frac{\mu}{\ell}\right) \ , \qquad c_L = \frac{3}{2G}\left(1 + \frac{\mu}{\ell}\right)$$

  Positive central charge requires $\mu\ell > 1$.

- Linearized excitations around AdS_3:

  (Li, Song & Strominger):

$$(E_0, s) \ = \ (2,2) \ , \quad (E_0, s) = (2,-2) \ ,$$
$$(E_0, s) \ = \ (\mu\ell + 1, 2) \quad SO(2,2) \text{ labels used.}$$

A dilemma:

- Sign of G as above gives positive BH mass but negative graviton mass.

- Letting $G \to -G$, gives positive graviton mass but negative BH mass.

  "the correct wrong sign" which Deser and collaborators have used for a long time.

Li, Song & Strominger proposal:

(arXive:0801.4566)

Consider the chiral point defined by: $\boxed{\mu\ell = 1}$

Then the wave function of the unwanted state coincides with that of $(E_0, s) = (2, 2)$ state, which can be gauged away!

Also: $\boxed{c_L = 0\ , c_R = \frac{3}{G}}$ called chiral gravity.

Desirable to solve the problem encountered by Witten & Maloney in physical interpretation of the partition function of the theory without the CS term.

Trouble:

- Carlip, Deser, Waldron & Wise get negative energy states even at the chiral point. (arXiv:0803.3998)

- Grumiller & Johansson (arXiv:0805.2610) clarified this by exhibiting a negative energy logarithmic mode, relaxed Brown-Henneaux boundary conditions consistent with AdS_3 asymptotics.

  New state has definite spin $s = 2$ but not energy eigenstate, leading to a logarithmic CFT. Thus problems with instability in the bulk, and non-chirality in the CFT dual.

- A challenging open problem. A consistent truncation to a CFT needs to be shown. Another hope lies in a better understanding of nonlinear equations.

  A non-perturbative positive energy theorem would do nicely!

- Preliminary results (work in progress) show that the supergravity version has the same dilemmas the bosonic theory has.

- It may be necessary to embed the model in string theory.

The model (Deser & Kay, 1983; Deser, 1984):

$$
\begin{aligned}
e^{-1}\mathcal{L} \;=\;& R - 2\varepsilon^{\mu\nu\rho}\bar{\psi}_\mu D_\nu(\omega)\psi_\rho \\
& +2m^2 - m\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu \\
& -\frac{1}{4}\mu^{-1}\,\varepsilon^{\mu\nu\rho}\left(R_{\mu\nu}{}^{ab}\omega_{\rho ab} + \tfrac{2}{3}\omega_\mu^{ab}\omega_{\nu b}{}^{c}\omega_{\rho ca}\right) \\
& -\mu^{-1}\bar{R}^\mu\gamma_\nu\gamma_\mu R^\nu
\end{aligned}
$$

- Overall factor $1/16\pi G$ is set to one.
- Definition: $R^\mu = \varepsilon^{\mu\nu\rho}D_\nu(\omega)\psi_\rho$

Supersymmetry:

$$\begin{aligned} \delta e_\mu^a &= \bar{\epsilon}\gamma^a\psi_\mu \\ \delta\psi_\mu &= D_\mu(\omega)\epsilon - \tfrac{1}{2}m\gamma_\mu\epsilon \end{aligned}$$

Why is there no μ dependence in susy transformations?

$$[\delta_1, \delta_2]\psi_\mu = -\tfrac{1}{4}\xi^\lambda\gamma_\mu\gamma_\lambda\gamma^\rho(R_\rho + \tfrac{1}{2}m\gamma_{\rho\sigma}\psi^\sigma) + \cdots$$

The missing $\mu^{-1}C_\rho$ term can be added freely since $\gamma^\rho C_\rho = 0$.

Field equations:

$$\mathcal{G}_{\mu\nu} + \mu^{-1} C_{\mu\nu} = 0$$
$$R^{\mu} + \tfrac{1}{2} m \gamma^{\mu\nu} \psi_{\nu} + \tfrac{1}{2}\mu^{-1} C^{\mu} = 0$$

where

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R - m^2 g_{\mu\nu}$$

$$\begin{aligned} C_{\mu\nu} &= \varepsilon_{\mu}{}^{\rho\sigma} \nabla_{\rho} (R_{\sigma\nu} - \tfrac{1}{4} g_{\sigma\nu} R) \\ C^{\mu} &= \gamma^{\rho}\gamma^{\mu\nu} D_{\nu} R_{\rho} - \epsilon^{\mu\nu\rho} \left(R_{\rho\sigma} - \tfrac{1}{4} g_{\rho\sigma} R \right) \gamma^{\sigma} \psi_{\nu} \end{aligned}$$

Cotton tensor and “Cottino” vector spinor.

The general supersymmetric solution:

- The vector

$$K^\mu = \bar{\epsilon}\gamma^\mu\epsilon \qquad \Rightarrow \qquad K^\mu K_\mu = 0$$

since $\gamma_{\mu(\alpha\beta}\gamma^\mu{}_{\gamma)\delta} = 0$.

- Killing spinor equation

$$D_\mu\epsilon - \tfrac{1}{2}\gamma_\mu\epsilon = 0$$

$$\Rightarrow \qquad K^\mu\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$$

Thus, K^μ is a null Killing vector.

The integrability condition for $\hat{D}_\mu \epsilon = 0$:

$$\mathcal{G}_{\mu\nu}\,\gamma^\nu \epsilon = 0$$

$$\bar{\epsilon}\times \quad \Rightarrow \qquad \mathcal{G}_{\mu\nu}\,K^\nu = 0$$

$$\bar{\epsilon}\gamma^\rho\times \quad \Rightarrow \qquad \epsilon^{\mu\nu\rho}\,K_\nu\,\mathcal{G}_{\rho\sigma} = 0$$

The last two eqs exhaust all the content of the integrability condition.

Using the existence of the null Killing vector helps us in parameterizing the metric accordingly.

We then solve the field equation, and show that all the integrability conditions are satisfied.

The final result:

$$\mu^2 \neq 1 : \; ds^2 = d\rho^2 + 2e^{2\rho}\, du\, dv + e^{(1-\mu)\rho}\, f(u)\, du^2$$

$$\mu = 1 : \; ds^2 = d\rho^2 + 2e^{2\rho}\, du\, dv + \rho\, f(u)\, du^2$$

$$\mu = -1 : \; ds^2 = d\rho^2 + 2e^{2\rho}\, du\, dv + \rho\, e^{2\rho}\, f(u)\, du^2$$

- In terms of $z = e^{\rho}$ the $\mu = \pm 1$ solutions have logarithmic singularities.

- For $f(u) = 0$ solution has $N = 2$, while for $f(u) \neq 0$ it has $N = 1$ supersymmetry.

- Special cases found by:
  - Dereli & Sarioglu , 2000
  - Olmez, Sarioglu & Tekin, 2005
  - Carlip, Deser, Waldron & Wise, 2008.

- Our supersymmetric solutions exhibit the same number of local degrees of freedom for all values of the CS coupling constant μ, including the chiral values $\mu = \pm 1$.

- However, our metric is written in terms of coordinates on the Poincaré patch of AdS_3. Working in global coordinate system in which the AdS_3 metric is:

$$ds^2(\text{AdS}) = -(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\phi^2$$

we find that, no matter what choice we make for $f(u)$, our metric will contain singularities at $r = \infty$ when $\sin \frac{1}{2}(\tau + \phi)$ or $\sin \frac{1}{2}(\tau - \phi)$ vanishes.

In all three cases, the Cotton tensor can be written simply in terms of the Killing vector K:

$$\mu^2 \neq 1: \ C_{\mu\nu} = \tfrac{1}{2}\mu(1-\mu^2) f(u)\, e^{-(3+\mu)\rho}\, K_\mu\, K_\nu$$

$$\mu = +1: \ C_{\mu\nu} = f(u)\, e^{-4\rho}\, K_\mu\, K_\nu\,,$$

$$\mu = -1: \ C_{\mu\nu} = f(u)\, e^{-2\rho}\, K_\mu\, K_\nu$$

Thus conformally non-flat (hence inequivalent to AdS_3) whenever $f(u) \neq 0$.

General solution with null Killing vector

More general class of solutions by only assuming a null Killing vector, without the further property $\nabla_\mu K_\nu = -\epsilon_{\mu\nu\rho} K^\rho$ that follows from susy?

We have found the general solutions:

$$\mu^2 \neq 1: \; ds^2 = d\rho^2 + 2e^{2\rho}\, du\, dv + e^{(1\pm\mu)\rho}\, f(u)\, du^2$$

$$\mu^2 = 1: \; ds^2 = d\rho^2 + 2e^{2\rho}\, du\, dv + \rho\, f(u)\, du^2$$

$$or \;\; ds^2 = d\rho^2 + 2e^{2\rho}\, du\, dv + \rho\, e^{2\rho}\, f(u)\, du^2$$

Either supersymmetric as they stand, or supersymmetric if the opposite sign choice for the parameter μ in theory were taken.

Kerr-Schild form and vanishing quantum corrections

We find that all the solutions we have constructed can be written in Kerr-Schild form:

$$g_{\mu\nu} = g_{\mu\nu}(\text{AdS}) + s(u,\rho)\, K_\mu\, K_\nu$$

where the function $s(u,\rho)$ is given by

$$s(u,\rho) = h(u,\rho)\, e^{-4\rho}$$

The metric $g_{\mu\nu}(\text{AdS})$ is just the AdS_3 metric on the Poincaré patch, written in the form

$$ds^2(\text{AdS}) = d\rho^2 + 2e^{2\rho}\, du\, dv$$

A consequence of the Kerr-Schild form:

Our supersymmetric solutions will still solve the quantum corrected equations, possibly with a shifted value for the Chern-Simons coupling constant μ, since there will always result at least one factor of $K_\mu K^\mu$ which vanishes assuming that:

The r.h.s. in the EOM is replaced by symm. conserved tensor constructed from a finite number of polynomial terms, each of which is a monomial in the Riemann tensor and its covariant derivatives.

See: Coley, Gibbons, Hervik & Pope, 2008 "Metrics with vanishing quantum corrections".

Positivity of the Energy?

Let us try Witten's approach, using Killing spinors.

- Take the supercurrent for local supersymmetry to be:

$$J^{\mu}(\epsilon_1) = \nabla_{\nu}(\bar{\epsilon}_1 \, \gamma^{\mu\nu\rho} \, \psi_{\rho})$$

- Using its variation, define the quantity:

$$\begin{aligned} X &\equiv \int_{\Sigma} \delta_{\epsilon_2} \, J^{\mu}(\epsilon_1) d\Sigma_{\mu} \\ &= \int_{\Sigma} \nabla_{\nu}(\bar{\epsilon}_1 \, \gamma^{\mu\nu\rho} \, \hat{\nabla}_{\rho}\epsilon_2) d\Sigma_{\mu} \end{aligned}$$

- After some algebra, find:

$$X = \int_{\Sigma} \left(\hat{\nabla}_\nu \bar{\epsilon}_1 \, \gamma^{\mu\nu\rho} \, \hat{\nabla}_\rho \epsilon_2 + \frac{1}{2} \mathcal{G}_\nu{}^\mu \bar{\epsilon}_1 \gamma^\nu \epsilon_2 \right) d\Sigma_\mu$$

- Is X related to the energy? Consider a "deformed" metric that satisfies the Einstein equations

$$\mathcal{G}_{\mu\nu} = 8\pi G \, T_{\mu\nu}$$

  and that is close to a "vacuum" background metric satisfying $\mathcal{G}_{\mu\nu} = 0$. The vacuum is assumed to admit a Killing spinor ϵ, so $\hat{\nabla}_\mu \epsilon = 0$ in the background.

Taking $\epsilon_1 = \epsilon_2 = \epsilon$ in the expression for X, evaluated in the deformed metric, we see that the first term is quadratically small since $\widehat{\nabla}\epsilon$ itself is linearly small. Thus for a sufficiently small deformation, X can be taken to be given by

$$X = 4\pi G \int_{\Sigma} T_{\mu\nu} K^{\nu} \, d\Sigma^{\mu}$$

Indeed, Abbot-Deser style energy formula.

The standard way of proceeding is now to impose on the spinor field ϵ the Witten condition:

$$\gamma^i \hat{\nabla}_i \epsilon = 0$$

where te hindex i labels quantities in the surface Σ. Subject to the Witten condition, the first term in X is negative semi-definite, as desired. However, the second term is problematic. Using the EOM we have

$$E[K] \geq \frac{1}{8\pi\mu G} \int_{\Sigma} C_{\nu}{}^{\mu} \xi^{\nu} \, d\Sigma_{\mu}$$

However, because $C_{\nu}{}^{\mu}$ is 3rd order in derivatives of the metric, it will in general have no definite sign.

Thus:

Our Witten identity is incapable of providing an answer to the question of whether excitations around an AdS_3 background must always have positive energy.

An important open problem:

Relaxing the condition of solutions "close to a background metric satisfying $\mathcal{G}_{\mu\nu} = 0$", try to revamp the Nester-Witten energy formula to prove a positive energy theorem.

Ultimate problem:

Find the solution space of the theory and solve the theory exactly.