A FLAVOUR-SYMMETRIC PERSPECTIVE ON NEUTRINO MIXING

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"Tri-Bimaximal Lepton Mixing and the Neutrino Oscillation Data" P.F. Harrison, D. H. Perkins, W. G. Scott, Phys. Lett. B. 530 (2002) 167. hep-ph/0202074 (see also: HPS hep-ph/9904297)

$$\left(|U_{l\nu}|^2\right) = \begin{array}{c|ccc|} & \nu_1 & \nu_2 & \nu_3 \\ e & 2/3 & 1/3 & 0 \\ \mu & 1/6 & 1/3 & 1/2 \\ \tau & 1/6 & 1/3 & 1/2 \end{array}$$

NOW OFFICIALLY A "FAMOUS" PAPER (> 250 CITES).

"A TREMENDOUS ACHIEVEMENT!"
T. D. LEE AT CERN - 30 AUG 2007
(CERN indico video min. 42!!)

OF COURSE IT IS ACTUALLY THE EXPERIMENTS WHICH ARE TREMENDOUS!

OUTLINE OF TODAYS TALK: (emphasis on Flavour-Symmetry)

1) "Plaquette Invariants and the Flavour-Symmetric…" P.F. Harrison, D. R. J. Roythorne, and W. G. Scott, Phys. Lett. B 657 (2007) 210. arXive:0709.1439 [hep-ph]
2)"Real Invariant Matrices and Flavour-Symmetric…" P.F. Harrison, W. G. Scott and T. J. Weiler, Phys. Lett. B 641 (2006) 372. hep-ph/0607336
3)"Simplified Unitarity Triangles for the Lepton Sector…" J. D. Bjorken, P.F. Harrison, and W. G. Scott, Phys. Rev D 74 (2006) 073012. hep-ph0511201
4)"Covariant Extremisation of Flavour-Symmetric Jartlskog Invariants…" P.F. Harrison, and W. G. Scott Phys. Lett. B 628 (2005) 93. hep-ph/0508012
5) "The Simplest Neutririo Mass Matrix" P. F. Harrison and W. G. Scott Phys Lett. B B594 (2004) 324. hep-ph/0403278.

……..

"Review" of past few years 2004-2007 of HS…

T. D. LEE LECTURE AT CERN 30 AUG 2007

CERN video: http://indico.cern.ch/conferenceDisplay.py?confId=19674 (min. 42)

WE DID “ACHIEVE” SOMETHING HOWEVER:
WE PREDICTED TWO SM+ PARAMETERS!!:

$$\boxed{U_{e2} = U_{\mu 2} = 1/3}$$

Tri-Bi-Maximal
(HPS 1999/2002)

$$\begin{array}{c|ccc} & \nu_1 & \nu_2 & \nu_3 \\ e & 2/3 & 1/3 & 0 \\ \mu & 1/6 & 1/3 & 1/2 \\ \tau & 1/6 & 1/3 & 1/2 \end{array}$$

CHOOZ EXPT.
SAYS < 0.03
(not HS/HPS!!)

Tri-Maximal Mixing
(HS/HPS 1994/1995)

$$\begin{array}{c|ccc} & \nu_1 & \nu_2 & \nu_3 \\ e & 1/3 & 1/3 & 1/3 \\ \mu & 1/3 & 1/3 & 1/3 \\ \tau & 1/3 & 1/3 & 1/3 \end{array}$$

IMMEDIATELY GENERALISES

via Tri-***Phi***-Maximal
& Tri-***Chi***-Maximal
(HS 2002)

There was never a prediction from HPS/HS of exact Ue3≡0!

HS/BHS (2002-2006)

$$\begin{array}{c|ccc} & \nu_1 & \nu_2 & \nu_3 \\ e & * & 1/3 & * \\ \mu & * & 1/3 & * \\ \tau & * & 1/3 & * \end{array}$$

“ν_2-Trimaximal Mixing”

“S3 Group Mixing”

“Magic-Square Mixing”

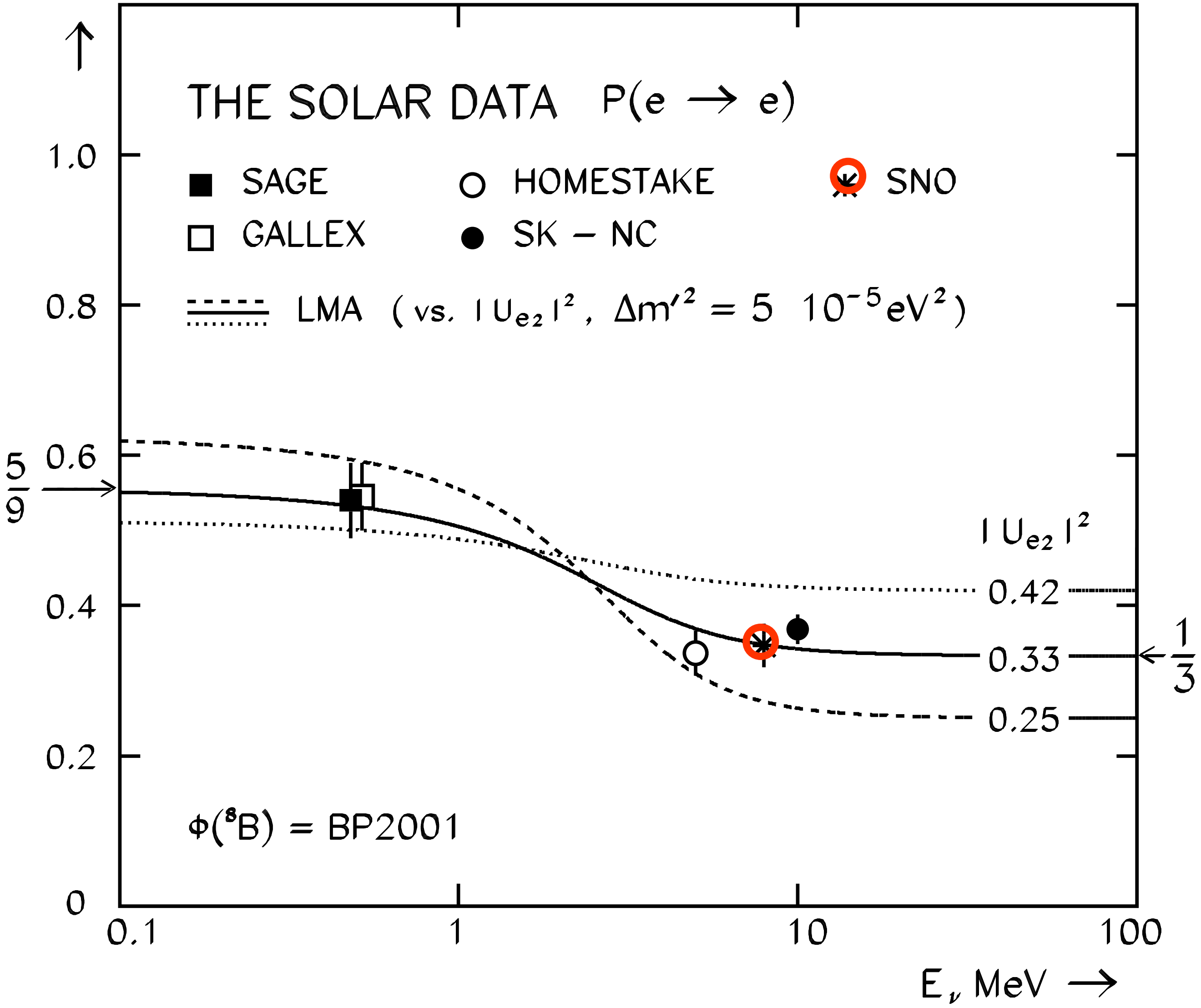
“Tri-χφ-Maximal”

Please not just “tri-maximal”!!

Solar Data

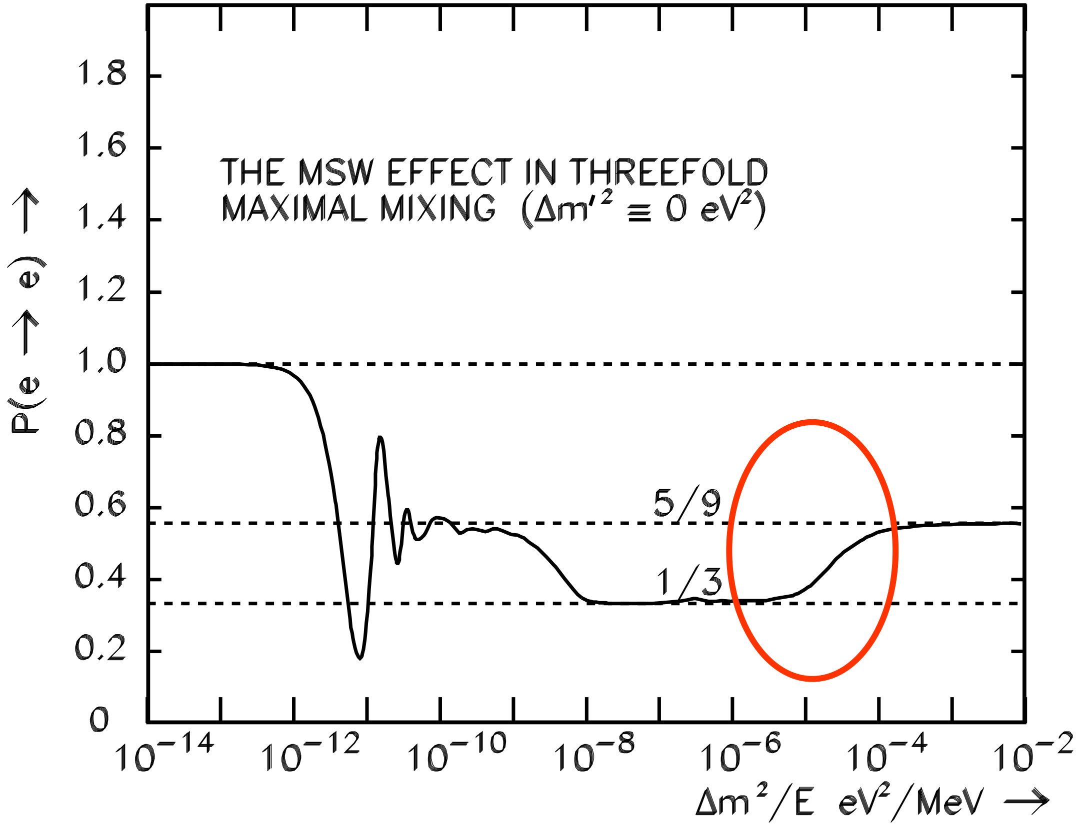
SNO point $CC/NC = 0.35 \pm 0.03$ is my naive average of Salt + No Salt - ignoring corr. systs. and ^{8}B spect. dist.

HPS PLB 530 (2002) 167 hep-ph/0202074; see also PLB 374 (1996) 111 hep-ph/9601346



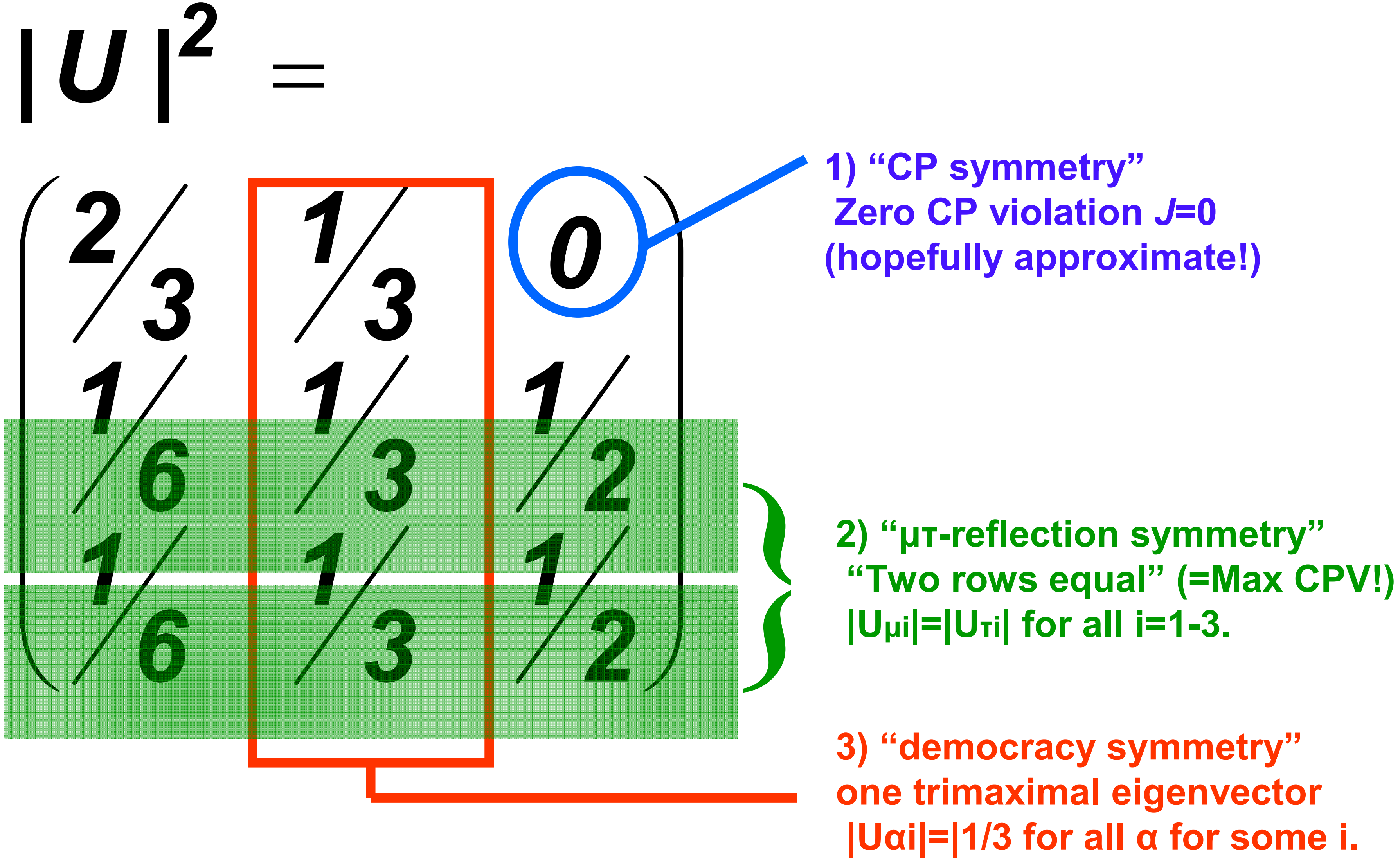
THE “5/9-1/3-5/9” BATHTUB

Fig. 3 HPS PLB 374 (1996) 111. hep-ph/9601346



PREDICTED IN TRIMAX. MIX. !!!!

Symmetries of TriBimaximal Mixing:



$$|U|^2 = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix}$$

1) “CP symmetry”
Zero CP violation J=0
(hopefully approximate!)

2) “μτ-reflection symmetry”
“Two rows equal” (=Max CPV!)
$|U_{\mu i}|=|U_{\tau i}|$ for all i=1-3.

3) “democracy symmetry”
one trimaximal eigenvector
$|U_{\alpha i}|=|1/3$ for all α for some i.

HPS "Derivation" of TriBimaximal Mixing:

Harrison, Perkins, Scott, Phys. Lett. B. 530 (2002) 167. hep-ph/0202074

In the "circulant basis": *

3 x 3 circulant (by definition of the * basis)

2 x 2 circulant (determines the physics)

$$a = m_e^2/3 + m_\mu^2/3 + m_\tau^2/3$$

$$b = m_e^2/3 + m_\mu^2\omega/3 + m_\tau^2\overline{\omega}/3$$

$$\overline{b} = m_e^2/3 + m_\mu^2\overline{\omega}/3 + m_\tau^2\omega/3$$

$$M \rightarrow MM^{\mp}$$

$$M_l M_l^{\dagger} = \begin{array}{c|ccc} & * & * & * \\ \hline * & a & b & \overline{b} \\ * & \overline{b} & a & b \\ * & b & \overline{b} & a \end{array}$$

$$M_\nu M_\nu^{\dagger} = \begin{array}{c|ccc} & * & * & * \\ \hline * & x & 0 & y \\ * & 0 & z & 0 \\ * & y & 0 & x \end{array}$$

A popular choice:

$$M_l = \begin{array}{c|ccc} * & m_e & m_\mu & m_\tau \\ * & m_e & m_\mu\omega & m_\tau\overline{\omega} \\ * & m_e & m_\mu\overline{\omega} & m_\tau\omega \end{array}$$

$$U_l^{\mp} M_l M_l^{\mp} U_l = \mathrm{diag}^2\{m_e\ m_\mu\ m_\tau\}$$

$$U_\nu^{\mp} M_\nu M_\nu^{\mp} U_\nu = \mathrm{diag}^2\{m_1\ \ m_2\ \ m_3\}$$

$$U = U_l^{\mp} U_\nu$$

$$\begin{array}{c|ccc} & * & * & * \\ \hline e & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \mu & \frac{\omega}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\varpi}{\sqrt{3}} \\ \tau & \frac{\varpi}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} \end{array} \quad \begin{array}{c|ccc} & \nu_1 & \nu_2 & \nu_3 \\ \hline * & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ * & 0 & 1 & 0 \\ * & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{array} \Rightarrow \begin{array}{c|ccc} & \nu_1 & \nu_2 & \nu_3 \\ \hline e & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \mu & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{2}} \\ \tau & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{2}} \end{array}$$

YES – YOU'VE SEEN THESE NUMBERS BEFORE SOMEWHERE!

e.g.

M = 0

SUBSET

OF

$j_1 = 1 \otimes j_2 = 1$

CLEBSCH-

GORDAN

COEFFS.

COULD

PERHAPS BE

A USEFUL

REMARK ?!!

		J	2	0	1
m_1	m_2	M	0	0	0
0	0		$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$	0
+1	−1		$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{2}}$
−1	+1		$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{2}}$

See: J. D. Bjorken, P. F. Harrison and W.G. Scott. hep-ph/0511201

"Symmetries and Generalisations of Tri-Bimaxiaml Mixing" P.F. Harrison, and W. G. Scott
Phys. Lett. B 535 (2002) 163. hep-ph/0203029

"Tri-χ-Maximal Mixing"

Exact μτ - Refl. Symm., J≠0

$$\begin{pmatrix} \sqrt{\frac{2}{3}}c_\chi & \frac{1}{\sqrt{3}} & -i\sqrt{\frac{2}{3}}s_\chi \\ -\frac{c_\chi}{\sqrt{6}}+i\frac{s_\chi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{c_\chi}{\sqrt{2}}+i\frac{s_\chi}{\sqrt{6}} \\ -\frac{c_\chi}{\sqrt{6}}-i\frac{s_\chi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{c_\chi}{\sqrt{2}}+i\frac{s_\chi}{\sqrt{6}} \end{pmatrix}$$

Φ→0

"Tri-Φ-Maximal Mixing"

J=0, Break μτ-Symmetry

$$\begin{pmatrix} \sqrt{\frac{2}{3}}c_\varphi & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}}s_\varphi \\ -\frac{c_\varphi}{\sqrt{6}}-\frac{s_\varphi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{c_\varphi}{\sqrt{2}}+\frac{s_\varphi}{\sqrt{6}} \\ -\frac{c_\varphi}{\sqrt{6}}+\frac{s_\varphi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{c_\varphi}{\sqrt{2}}+\frac{s_\varphi}{\sqrt{6}} \end{pmatrix}$$

Χ→0

"Permutation Symmetry, Tri-Bimaximal Mixing and the S3 Group ..." P.F. Harrison, and W. G. Scott
Phys. Lett. B 557 (2003) 76. hep-ph/0302025

"ν_2-Trimaximal Mixing"

"Tri-φχ-maximal mixing", "S3 group mixing"
"Magic-square mixing", "BHS-mixing"… .

$$\begin{pmatrix} \sqrt{\frac{2}{3}}c_\chi c_\varphi+i\sqrt{\frac{2}{3}}s_\chi s_\varphi & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}}c_\chi s_\varphi-i\sqrt{\frac{2}{3}}s_\chi c_\varphi \\ -\frac{c_\chi c_\varphi+is_\chi s_\varphi}{\sqrt{6}}-\frac{c_\chi s_\varphi-is_\chi c_\varphi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{c_\chi c_\varphi-is_\chi s_\varphi}{\sqrt{2}}+\frac{c_\chi s_\varphi+is_\chi c_\varphi}{\sqrt{6}} \\ -\frac{c_\chi c_\varphi+is_\chi s_\varphi}{\sqrt{6}}+\frac{c_\chi s_\varphi-is_\chi c_\varphi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{c_\chi c_\varphi-is_\chi s_\varphi}{\sqrt{2}}+\frac{c_\chi s_\varphi+is_\chi c_\varphi}{\sqrt{6}} \end{pmatrix}$$

$s_\chi = \sin\chi$
$c_\chi = \cos\chi$
$s_\varphi = \sin\varphi$
$c_\varphi = \cos\varphi$

Nature Plays Sudoku !!

$$\left\{\begin{array}{l}\text{“}\nu_2\text{-Trimaximal Mixing”}\\ =\text{“Magic-Square”/”S3 Group Mixing”}\\ =\text{“Democracy Symmetry”}\end{array}\right.$$

“Permutation Symmetry, Tri-Bimaximal Mixing and the S3 Group ...” P.F. Harrison, and W. G. Scott Phys. Lett. B 557 (2003) 76. hep-ph/0302025

Experiment tells us that the neutrino mass matrix² $M_\nu M_\nu^\dagger$ in the (charged-lepton) flavour basis can be written as a 3 x 3 Magic Square !!

All row/column sums equal !!

Symmetric Group S3
(natural representation):

$$I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$P(123)=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\quad P(321)=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$P(12)=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad P(23)=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad P(31)=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

The most general such (hermitian) matrix may be constructed as an “S3 Group Matrix”

in the natural representation of the S3 group **ring**

$$M_\nu M_\nu^\dagger = \quad \text{"even"} \quad + \quad \text{"odd"}$$

$$= aI + bP(123) + b^*P(321) \quad + \quad xP(23) + yP(31) + zP(12)$$

$$\left\{\begin{array}{l}\tan 2\chi = \sqrt{6}\,(Imb)/(x^2+y^2+z^2-xy-yz-zx)\\ \tan 2\varphi = \sqrt{3}\,(z-y)/(x+y-2x)\end{array}\right.$$

$$= \begin{array}{c|ccc} & \nu_e & \nu_\mu & \nu_\tau \\ \hline \nu_e & a & b & b^* \\ \nu_\mu & b^* & a & b \\ \nu_\tau & b & b^* & a \end{array} \quad + \quad \begin{array}{c|ccc} & \nu_e & \nu_\mu & \nu_\tau \\ \hline \nu_e & x & z & y \\ \nu_\mu & z & y & x \\ \nu_\tau & y & x & z \end{array}$$

“circulant” “retro-circulant”

Any “S3 Group Matrix” clearly has (at least) one trimaximal eigenvector:

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

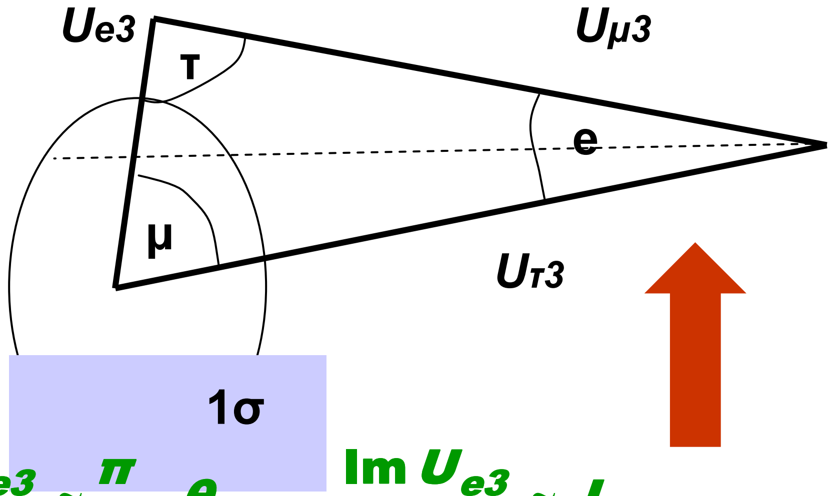
Simplified Unitarity Traingles in the Lepton Sector

J. D. Bjorken, P.F. Harrison, and W. G. Scott, Phys. Rev D 74 (2006) 073012. hep-ph0511201

"BHS" Mixing = "ν₂-Trimaximal"

$$\begin{array}{c|ccc} & \nu_1 & \nu_2 & \nu_3 \\ \hline e & \frac{2}{\sqrt{6}}C & \frac{1}{\sqrt{3}} & U_{e3} \\ \mu & -\frac{1}{\sqrt{6}}C & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}C-\frac{1}{2}U_{e3} \\ \tau & -\frac{1}{\sqrt{6}}C & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}}C-\frac{1}{2}U_{e3} \end{array}$$

"ν2.ν3"="the ν1-triangle"



$$\frac{\mathrm{Re}\,U_{e3}}{\sqrt{2}} \approx \frac{\pi}{4} - \theta_{23} \qquad \frac{\mathrm{Im}\,U_{e3}}{3\sqrt{2}} \approx J_{CP}$$

The Matrix* of UT angles:

$$\Phi = \begin{array}{c|ccc} & \nu_1 & \nu_2 & \nu_3 \\ \hline e & \varphi_{e1} & \varphi_{e2} & \varphi_{e3} \\ \mu & \varphi_{\mu1} & \varphi_{\mu2} & \varphi_{\mu3} \\ \tau & \varphi_{\tau1} & \varphi_{\tau2} & \varphi_{\tau3} \end{array}$$

***Footnote [42] hep-ph/0511201**
Note the natural "complementary" labelling of angles and triangles

Each angle $\Phi_{\alpha i}$ appears in one row-based triangle and one column-based triangle

"Simplified Unitarity Triangles for the Lepton Sector…"

J. D. Bjorken, P.F. Harrison, and W. G. Scott, Phys. Rev D 74 (2006) 073012. **hep-ph/0511201**

From the Plaquette Products:

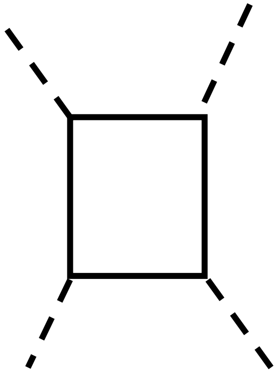
$$\Pi_{l\nu} := U_{l-1\ \nu-1} U^{*}_{l-1\ \nu+1} U_{l+1\ \nu+1} U^{*}_{l+1\ \nu-1} \qquad (\mathrm{mod}\ \ 3)$$

$$\Pi_{l\nu} := -K_{l\nu} + \ i\, J$$

$$\begin{array}{cc} & \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{array}{c} e \\ \mu \\ \tau \end{array} & \begin{pmatrix} U_{l-1\ \nu-1} & \times & U^{*}_{l-1\ \nu+1} \\ \times & \Pi_{l\nu} & \times \\ U^{*}_{l+1\ \nu-1} & \times & U_{l+1\ \nu+1} \end{pmatrix} \end{array}$$

Form the Matrix of Plaquette Products:

$$\Pi = \begin{array}{cc} & \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{array}{c} e \\ \mu \\ \tau \end{array} & \begin{pmatrix} U_{\mu 2}U_{\tau 3}U^{*}_{\mu 3}U^{*}_{\tau 2} & U_{\mu 3}U_{\tau 1}U^{*}_{\mu 1}U^{*}_{\tau 3} & U_{\mu 1}U_{\tau 2}U^{*}_{\mu 2}U^{*}_{\tau 1} \\ U_{e3}U_{\tau 2}U^{*}_{e2}U^{*}_{\tau 3} & U_{e1}U_{\tau 3}U^{*}_{e3}U^{*}_{\tau 1} & U_{e2}U_{\tau 1}U^{*}_{e1}U^{*}_{\tau 2} \\ U_{e2}U_{\mu 3}U^{*}_{e3}U^{*}_{\mu 2} & U_{e3}U_{\mu 1}U^{*}_{e1}U^{*}_{\mu 3} & U_{e1}U_{\mu 2}U^{*}_{e2}U^{*}_{\mu 1} \end{pmatrix} \end{array}$$



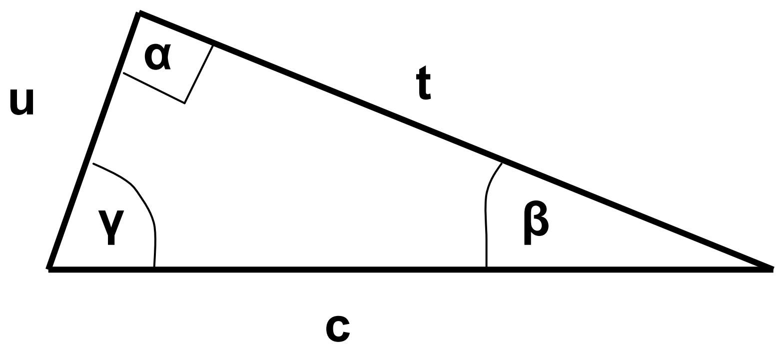
We define the Matrix of UT Angles:*

***Footnote [42] hep-ph/0511201**

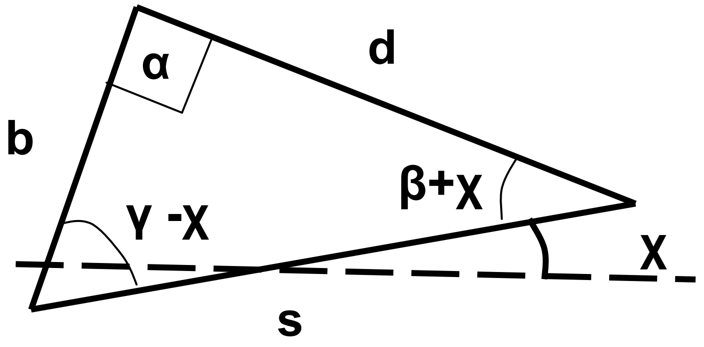
$$\Phi = \begin{array}{cc} & \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{array}{c} e \\ \mu \\ \tau \end{array} & \begin{pmatrix} \varphi_{e1} & \varphi_{e2} & \varphi_{e3} \\ \varphi_{\mu 1} & \varphi_{\mu 2} & \varphi_{\mu 3} \\ \varphi_{\tau 1} & \varphi_{\tau 2} & \varphi_{\tau 3} \end{pmatrix} \end{array} = \begin{array}{ccc} & \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} & \\ \begin{array}{c} e \\ \mu \\ \tau \end{array} & \begin{pmatrix} \mathrm{Arg}\text{ - }\Pi_{e1} & \mathrm{Arg}\text{ - }\Pi_{e2} & \mathrm{Arg}\text{ - }\Pi_{e3} \\ \mathrm{Arg}\text{ - }\Pi_{\mu 1} & \mathrm{Arg}\text{ - }\Pi_{\mu 2} & \mathrm{Arg}\text{ - }\Pi_{\mu 3} \\ \mathrm{Arg}\text{ - }\Pi_{\tau 1} & \mathrm{Arg}\text{ - }\Pi_{\tau 2} & \mathrm{Arg}\text{ - }\Pi_{\tau 3} \end{pmatrix} & \begin{array}{c} |\pi \\ |\pi \\ |\pi \end{array} \\ & \begin{array}{ccc} \overline{\quad\pi\quad} & \overline{\quad\pi\quad} & \overline{\quad\pi\quad} \end{array} & \end{array}$$

UNITARITY TRIANGLES IN THE QUARK SECTOR

"d.b"="the s-triangle"



"t.u"="the c-triangle"



($\chi \approx 1^o$ in SM - see e.g. F. Muheim "Flavour in the Era of LHC" HEP Forum 21 June 2007)

THE MATRIX OF UNITARITY TRIANGLES IN THE QUARK SECTPR

$$\Phi = \begin{array}{cc} & \begin{array}{ccc} d & s & b \end{array} \\ \begin{array}{c} u \\ c \\ t \end{array} & \begin{pmatrix} \varphi_{ud}=\beta_s \approx \chi & \varphi_{us}=\beta & \varphi_{ub} \\ \varphi_{cd}=\gamma-\chi & \varphi_{cs}=\alpha & \varphi_{cb}=\beta+\chi \\ \varphi_{td} & \varphi_{ts}=\gamma & \varphi_{tb} \end{pmatrix} \end{array} \approx \begin{array}{cc} & \begin{array}{ccc} d & s & b \end{array} \\ \begin{array}{c} u \\ c \\ t \\ \\ \end{array} & \begin{array}{c} \begin{pmatrix} \overset{CDF/D0 \approx 20^o!!!}{\boxed{1^o}} & 22^o & 157^o \\ 67^o & 90^o & 23^o \\ \underline{112^o} & \underline{68^o} & \underline{0(\lambda^4)} \end{pmatrix} \begin{array}{l} |180^o \\ |180^o \\ |180^o \end{array} \\ \begin{array}{ccc} 180^o & 180^o & 180^o \end{array} \end{array} \end{array}$$

Systematic "complemenatry" notation here is a big improvement on existing notations!!

EQUIVALENT INFO. TO CKM MATRIX !!

"Plaquette Invariants and the Flavour-Symmetric …"

P.F. Harrison, D. R. J. Roythorn, and W. G. Scott, Phys. Lett. B 657 (2007) 210. arXive:0709.1439 [hep-ph]

The Principles which guide us:

$$L = M_l \to M_l M_l^{\dagger}$$
$$N = M_\nu \to M_\nu M_\nu^{\dagger}$$

1) **Flavour Symmetry**: **A fundamental theory of flavour should be Flavour-Symmetric (ie. it should make no reference to explicit flavour indices).**

2) **Jarlskog Invariance**: **A fundamental theory should be weak-basis independent (i.e. it should make no reference to any preferred weak-basis).**

➡ Use Flavour-Symmetric Jarlskog Invariant variables!!

The Architypal example:
The Jarlskog CP-Invariant:

$$J = Im\,\Pi_{l\nu} = -i\,\frac{Det\ [L,N]}{2L_\Delta N_\Delta}$$

Independent, of plaquette choice *l*,ν
hence "Plaquette Invariant"

The Jarlskogian *J* is "odd-odd" under separate *l* and ν flavour permutations:

$$(\bar{1} \times \bar{1}) \quad \Leftarrow \quad S3_l \times S3_\nu$$

We define 6 New Flavour-Symmetric Jarlskog-Invariant mixing variables :

with odd/even symmetry under: $S3_l \times S3_\nu$ (functions only of mixing angles)

$$F = (\bar{1} \times \bar{1})^{(2)} = Det\ P = \frac{Det\ T}{L_\Delta N_\Delta} = 3(wz - xy)$$

spanning the $C3_l \times C3_\nu$ Invariant polynomial ring

$$G = (1 \times 1)^{(2)} = (Tr\ P.P^T - 1)/2 = (w + x + y + z)^2 + (w^2 + x^2 + y^2 + z^2) - (wz + xy)$$

$$C = (1 \times 1)^{(3)} \quad A = (\bar{1} \times \bar{1})^{(3)} \quad B = (\bar{1} \times 1)^{(3)} \quad D = (1 \times \bar{1})^{(3)}$$

An `elemental" set - not all independent, e,g, $J^2 = 1/108 - G/18 + 2C/27 - F^2/36$

Jarlskog Invariance: (Also known as Weak-Basis Invaraince)

In any "weak" ("gauge") basis the weak interaction is diagonal and universal (i.e proportional to the identity matrix)

We often seem to choose to blame the mixing on the "down" quarks!

But we could equally choose to blame it on the "up"-type quarks!

$$\overset{\text{weak basis}}{\begin{pmatrix} u & c & t \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}} \quad \underset{\mathrm{U(3)}}{\Longleftrightarrow} \quad \overset{\text{weak basis}}{\begin{pmatrix} u' & c' & t' \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} d \\ s \\ b \end{pmatrix}}$$

CC weak int.

Elsewhere in the Lagrangian: (i.e in the yukawa sector)

$$\begin{array}{lcl} M_u \text{ is diagonal} & \Longleftrightarrow & M_d \text{ is diagonal} \\ (M_d \text{ is non-diagonal}) & & (M_u \text{ is non-diagonal}) \end{array} \Bigg\} \text{ Mass}^2 \text{ Matrices}$$

All observables are Jarlskog Invariant: $m_u \quad m_d \quad m_t \quad |V_{ub}|^2$

e.g. masses, mixing angles: $m_e \quad m_\mu \quad m_3 \quad \theta_{13} \quad \delta \quad J$ etc.

Note that the Jarlskogian J is (moreover) also Flavour-Symmetric !!

FLAVOUR-SYMMETRIC
JARLSKOG INVARIANT MASS PARAMETERS

Charged-Leptons: **Mass Matrix:** $L := M_l \rightarrow M_l M_l^\dagger$

$$\left.\begin{array}{lll} L_1 := \mathrm{Tr}\ L & = & m_e + m_\mu + m_\tau \\ L_2 := \mathrm{Tr}\ L^2 & = & m_e^2 + m_\mu^2 + m_\tau^2 \\ L_3 := \mathrm{Tr}\ L^3 & = & m_e^3 + m_\mu^3 + m_\tau^3 \end{array}\right\} \quad \begin{array}{c} \{L_1\ \ L_2\ \ L_3\} \\ \cong \\ \{m_e\ \ m_\mu\ \ m_\tau\} \end{array}$$

Neutrinos: **Mass Matrix:** $N := M_\nu \rightarrow M_\nu M_\nu^\dagger$

$$\left.\begin{array}{lll} N_1 := \mathrm{Tr}\ N & = & m_1 + m_2 + m_3 \\ N_2 := \mathrm{Tr}\ N^2 & = & m_1^2 + m_2^2 + m_3^2 \\ N_3 := \mathrm{Tr}\ N^3 & = & m_1^3 + m_2^3 + m_3^3 \end{array}\right\} \quad \begin{array}{c} \{N_1\ N_2\ N_3\} \\ \cong \\ \{m_1\ \ m_2\ \ m_3\} \end{array}$$

THE CHARACTERISTIC EQUATION

e.g. For the Charged-Lepton Masses:

$$\lambda^3 - (\mathrm{Tr}\ L)\lambda^2 + (\mathrm{Pr}\ L)\lambda - (\mathrm{Det}\, L) = 0$$

where:

$$\left.\begin{aligned}
\mathrm{Tr}\ L &= m_e + m_\mu + m_\tau = L_1 \\
\mathrm{Pr}\ L &= m_e m_\mu + m_\mu m_\tau + m_\tau m_e = (L_1^2 - L_2)/2 \\
\mathrm{Det}\, L &= m_e m_\mu m_\tau = (L_1^3 - 3L_1 L_2 + 2L_3)/6
\end{aligned}\right\}$$

The Disciminant:

$$\left.\begin{aligned}
L_\Delta^2 &= L_2^3/2 - 3L_1^4 L_2/2 + 6L_1 L_2 L_3 \\
&\qquad - 7L_1^2 L_2^2/2 - 3L_3^2 - 4L_1^3 L_3/3 - L_1^6/6 \\
&= (m_e - m_\mu)^2 (m_\mu - m_\tau)^2 (m_\tau - m_e)^2
\end{aligned}\right\}$$

All are Flavour-Symmetric and Jarlskog Invariant!!

Flavour-Symmetric Mixing Observables…

2 x 2 of $S3_l \times S3_v$

P.F. Harrison, D. R. J. Roythorn, and W. G. Scott, Phys. Lett. B 657 (2007) 210. arXive:0709.1439 [hep-ph]

Six New FS Variables ("Plaquette Invariants")

A, B, C, D, F, G, **analogous to Jarlskog** J,
order (n) with odd/even symmetry
under $(S3_l \times S3_v)$ - scalar or pseudoscalar.

$$P = (|U|^2) = \begin{pmatrix} \frac{1}{3}+\cdots & \frac{1}{3}+w & \frac{1}{3}+x \\ \frac{1}{3}+\cdots & \frac{1}{3}+y & \frac{1}{3}+z \\ \frac{1}{3}+\cdots & \frac{1}{3}+\cdots & \frac{1}{3}+\cdots \end{pmatrix}$$

$$F = (\bar{1} \times \bar{1})^{(2)} = \text{Det}\, P = \frac{\text{Det}\, T}{L_\Delta N_\Delta} = 3(wz - xy)$$

$$G = (1 \times 1)^{(2)} = (\text{Tr}\, P.P^T - 1)/2 = (w+x+y+z)^2 + (w^2+x^2+y^2+z^2) - (wz+xy)$$

$$C = (1 \times 1)^{(3)} = 9(xyz + wyz + wxz + wxy) + \frac{9}{2}[xy(x+y) + wz(w+z)]$$

$$A = (\bar{1} \times \bar{1})^{(3)} = 2(w^3 - x^3 - y^3 + z^3) + 3[wx(w-x) + wy(w-y) + yz(z-y) + xz(z-x) + xy(w+z) - wz(x+y)] + \frac{3}{2}[wz(w+z) - xy(x+y)]$$

$$B = (\bar{1} \times 1)^{(3)} = 3\sqrt{3}\,[w^2x + wx^2 - y^2z - yz^2 + wxy + wxz - xyz - wyz + \frac{1}{2}(w^2z - wz^2 + yx^2 - y^2x)]$$

$$D = (1 \times \bar{1})^{(3)} = -3\sqrt{3}\,[w^2y + wy^2 - x^2z - xz^2 + wxy + wyz - xyz - wxz + \frac{1}{2}(w^2z - wz^2 + xy^2 - x^2y)]$$

B, D are
not $l \leftrightarrow v$
symmetric

Not all independent $A^2 + B^2 + C^2 + D^2 = F^2G/2 + G^3/2$ $\quad AC + BD = 3G^2F/4 + F^3/4$

Plaquette Invariance (= $C3_l \times C3_v$ Invariance)

$$P = \begin{pmatrix} \frac{1}{3} - w - x & \frac{1}{3} + w & \frac{1}{3} + x \\ \frac{1}{3} - y - z & \frac{1}{3} + y & \frac{1}{3} + z \\ \frac{1}{3} + w + x + y + z & \frac{1}{3} - w - y & \frac{1}{3} - x - z \end{pmatrix}$$

$$F/3 = wz - xy$$

$$F/3 = (y+z)(w+y) - y(w+x+y+z)$$

$$= yw + y^2 + zw + zy \quad - yw - xy - y^2 - yz$$

$$= wz - xy$$

$\Rightarrow$ **"PLAQUETTE INVARIANT"!!!**

Flavour-Symmetric Weak-Basis-Invariant Constraints on Mixing:

Democracy Symmetry
ie. one column=(1/3,1/3.1/3), iff:

$$F = 0 \qquad\qquad C = 0$$

“ μ–τ ” - Reflection Symmetry,
ie. two rows (or columns) equal, iff:

$$F = 0 \qquad\qquad A = 0$$

Tri-Bi-Maximal Mixing, iff:

$$F = C = A = J = 0$$

$$G = 1/6$$

Solving more generally for the P-matrix

in the limit F, A, C → 0 and 0 < G < 1/6, gives:

$$P \approx \begin{pmatrix} \frac{1}{3} + 2\sqrt{\frac{G}{6}} + \frac{2C}{9G} & \frac{1}{3} - \frac{4C}{9G} & \frac{1}{3} - 2\sqrt{\frac{G}{6}} + \frac{2C}{9G} \\ \frac{1}{3} - \sqrt{\frac{G}{6}} - \frac{F}{2\sqrt{6G}} + \frac{A}{3G} - \frac{C}{9G} & \frac{1}{3} + \frac{F}{\sqrt{6G}} + \frac{2C}{9G} & \frac{1}{3} + \sqrt{\frac{G}{6}} - \frac{F}{2\sqrt{6G}} - \frac{A}{3G} - \frac{C}{9G} \\ \frac{1}{3} - \sqrt{\frac{G}{6}} + \frac{F}{2\sqrt{6G}} - \frac{A}{3G} - \frac{C}{9G} & \frac{1}{3} - \frac{F}{\sqrt{6G}} + \frac{2C}{9G} & \frac{1}{3} + \sqrt{\frac{G}{6}} + \frac{F}{2\sqrt{6G}} + \frac{A}{3G} - \frac{C}{9G} \end{pmatrix}$$

Jarlskog *J* measures CP-violation (J=0 protects against violation of CP).

F **measures the acoplanarity of the P-vectors in the flavour space**
(F=0 => Det <$\mathcal{P}(\infty)$> = 0, i.e. protects distant source against flavour analysis)

G = 3<<$P_{ll}(\infty)$>>-1 **measures the flavour-averaged asymptotic survival prob….**

Ansatz	F	G	C	A	Symm.	18J	B	D
Tri-Bi-Max.	0	1/6	0	0	Dem., μτ, CP	0	0	1/12√3
Tri-Max. Mix.	0	0	0	0	Dem., μτ	1/6	0	0
Tri-χφ-Max.	0	-	0	-	Dem.(ocracy)	-	0	-
2 Rows Eq.	0	-	-	0	e.g. μ-τ	-	0	-
2 Cols. Eq.	0	-	-	0	e.g. 1-2	-	-	0
Alt.-Feruglio	0	-	(6G-1)/8	0	μτ, CP	0	0	-
Tri-χ-Max.	0	-	0	0	Dem., μτ	-	0	-
Tri-φ-Max.	0	1/6	0	-	Dem., CP	0	0	-
Orig. Bi-Max.	0	1/8	-1/32	0	CP, μ-τ,1-2	0	0	0
No Mixing	1	1	1	1	CP	0	0	0

Directly in Terms of Mass Matrices:

$$L = M_l \to M_l M_l^\dagger$$
$$N = M_\nu \to M_\nu M_\nu^\dagger$$

The Jarlskog Commutator: $C = -i[L, N]$

controls CP violation: $\mathrm{Det}\ C = \mathrm{Tr}\ C^3 / 3$

Generalised Jarlskog Commutators: $C_{mn} = -i[L^m, N^n]$

And Anti-Commutators: $A_{mn} = \{L^m, N^n\}$

The Matrix of Cubic Commutator Traces

$$C^{(3)} = \begin{pmatrix} \mathrm{Tr}\ C_{11}^3 & \mathrm{Tr}\ C_{11}^2 C_{12} & \mathrm{Tr}\ C_{11} C_{12}^2 \\ \mathrm{Tr}\ C_{11}^2 C_{21} & \mathrm{Tr}\ C_{11}^2 C_{22} & \mathrm{Tr}\ C_{21} C_{12}^2 \\ \mathrm{Tr}\ C_{11} C_{21}^2 & \mathrm{Tr}\ C_{12} C_{21}^2 & \mathrm{Tr}\ C_{11} C_{22}^2 \end{pmatrix}$$

The Matrix of Anti-Commutator Traces
(traces of mass-matrix products):

$$T = \frac{1}{2} \begin{pmatrix} \mathrm{Tr}\ A_{00} & \mathrm{Tr}\ A_{01} & \mathrm{Tr}\ A_{02} \\ \mathrm{Tr}\ A_{10} & \mathrm{Tr}\ A_{11} & \mathrm{Tr}\ A_{12} \\ \mathrm{Tr}\ A_{20} & \mathrm{Tr}\ A_{21} & \mathrm{Tr}\ A_{22} \end{pmatrix}$$

For example, *F*:

$$F = \mathrm{Det}\ P = 3(wz - xy) = \frac{\mathrm{Det}\ T}{L_\Delta N_\Delta} = \frac{\mathrm{Det}\ C^{(3)}}{(L_\Delta N_\Delta\ \mathrm{Det}^3\ C)}$$

In terms of Mass Matrices only

“Real Invariant Matrices and Flavour-Symmetric…”

P.F. Harrison, W. G. Scott and T. J. Weiler, Phys. Lett. B 641 (2006) 372. hep-ph/0607336

The “P-matrix”: $P_{l\nu} = P(W \to l\nu)$

$$P = (|U|^2) = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu 1 & \nu 2 & \nu 3 \end{array} \\ \begin{pmatrix} |U_{e1}|^2 & |U_{e2}|^2 & |U_{e3}|^2 \\ |U_{\mu 1}|^2 & |U_{\mu 1}|^2 & |U_{\mu 1}|^2 \\ |U_{\tau 1}|^2 & |U_{\tau 1}|^2 & |U_{\tau 1}|^2 \end{pmatrix} \end{array}$$

“T-matrix”

$$T = \begin{array}{c|ccc} & n=0 & n=1 & n=2 \\ \hline m=0 & 3 & N_1 & N_2 \\ m=1 & L_1 & Tr\, L^1 N^1 & Tr\, L^1 N^2 \\ m=2 & L_2 & Tr\, L^2 N^1 & Tr\, L^2 N^2 \end{array}$$

Moment Transform: $T_{mn} = Tr\, L^m N^n = m_l^m . P . m_\nu^n$

(invertible)

The “K-matrix”

$$K_{l\nu} = -Re\ \Pi_{l\nu}$$

$$\Pi_{l\nu} = U_{l-1\,\nu-1} U_{l+1\,\nu+1} U^*_{l-1\,\nu+1} U^*_{l+1\,\nu-1}$$

”permanent”

$$K_{l\nu} = (P_{l\nu} - \{P_{l-1\,\nu-1} P_{l+1\,\nu+1} + P_{l+1\,\nu-1} P_{l-1\,\nu+1}\})/2 \quad (\mathrm{mod}\ 3)$$

$$K = (K_{l\nu}) = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu 1 & \nu 2 & \nu 3 \end{array} \\ \begin{pmatrix} K_{e1} & K_{e2} & K_{e3} \\ K_{\mu 1} & K_{\mu 2} & K_{\mu 3} \\ K_{\tau 1} & K_{\tau 2} & K_{\tau 3} \end{pmatrix} \end{array}$$

“Q-matrix”

$$Q = \frac{1}{2} \begin{pmatrix} Tr\, [L,N]^2 & Tr\, [L,N][L,N^2] & Tr\, [L,N^2]^2 \\ Tr\, [L,N][L^2,N] & Tr\, [L,N][L^2,N^2] & Tr\, [L,N^2][L^2,N^2] \\ Tr\, [L^2,N]^2 & Tr\, [L^2,N][L^2,N^2] & Tr\, [L^2,N^2]^2 \end{pmatrix}$$

Moment Transform: $Q_{mn} = \Delta_l^T\ diag\ \Delta_l\ (diag\ \Sigma_l)^{m-1}\ K\ (diag\ \Sigma_l)^{n-1}\ diag\ \Delta_\nu\ \Delta_\nu$

(invertible)

Expressed as Traces

$$L_G = L_G(L_1, L_2, L_3) \quad etc.$$

entirely in terms of
Mass Matrices

Two quadratic variables *G,F*

$$2G+1 = \mathrm{Tr}\;\; P^T.P \;=\; \mathrm{Tr}\;\; T^T.L_G.T.N_G$$

$$6F = \text{-}\mathrm{Tr}\;\; P^T.\in.P.\in \;=\; \text{-}\mathrm{Tr}\;\; T^T.L_F.T.N_F$$

No *l* ↔ν asymmetric quadratic variables:

$$\mathrm{Tr}\;\; P^T.\in.P \;=\mathrm{Tr}\;\; P^T.P.\in = 0$$

$$\in \;=\; \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Two *l* ↔ν symmetric cubic variables *C,A*:

$$2C/3 - G/2 - 1/6 = -\mathrm{Tr}\;\; P^T.K \;=\; \text{-}\mathrm{Tr}\;\; T^T.L_C.Q.N_C$$

$$2A\text{-}2F = \mathrm{Tr}\;\; P^T.\in.K.\in \;= \mathrm{Tr}\;\; T^T.L_A.Q.N_A$$

Two *l* ↔ν asymmetric cubic variables *B,D*:

$$2B/\sqrt{3} = \mathrm{Tr}\;\; P^T.\in.K \;= \mathrm{Tr}\;\; T^T.L_A.Q.N_C$$

$$2D/\sqrt{3} = \mathrm{Tr}\;\; P^T.K.\in \;= \mathrm{Tr}\;\; T^T.L_C.Q.N_A$$

Expressed as Traces (cont.)

$L_G = L_G(L_1, L_2, L_3)$ etc.

entirely in terms of
Mass Matrices

The Mass-Polynomial Matrices Requd:

$$L_F = \frac{1}{L_\Delta}\begin{pmatrix} 0 & L_2 & -L_1 \\ -L_2 & 0 & 3 \\ L_1 & -3 & 0 \end{pmatrix}$$

Anti-symmetric Matrix

$$L_0 = 3 \qquad L_\Delta = \mathrm{Det}\, L_G^{-1}$$

$$L_G = \begin{pmatrix} 3 & L_1 & L_2 \\ L_1 & L_2 & L_3 \\ L_2 & L_3 & L_4 \end{pmatrix}^{-1}$$

Symmetric Matrix

$$L_C = \ldots$$

$$L_A = \ldots$$

$$L_4 = L_1^4/6 + 4L_1 L_3/3 + L_2^2/2 - L_1^2 L_2$$

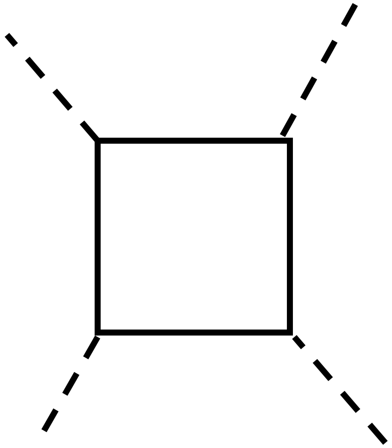
Flavour-Summed Loop Amplitudes

Usual Plaquette Product:

$$\begin{pmatrix} U_{l-1\ v-1} & \times & U^*_{l-1\ v+1} \\ \times & \Pi_{lv} & \times \\ U^*_{l+1\ v-1} & \times & U_{l+1\ v+1} \end{pmatrix}$$

$$\Pi_{lv} := -K_{lv} + iJ$$

4-Plaquette



$$\Pi_{lv} := U_{l-1\ v-1} U^*_{l-1\ v+1} U_{l+1\ v+1} U^*_{l+1\ v-1}$$

$$\sum_{lv} \Pi_{lv} = \underline{(G-1)/2 + 9i\ J} \qquad \sum_{lv} K_{lv} = (1-G)/2$$

Hexaplaquette Product:

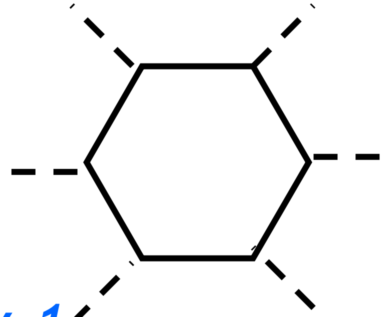
even

$$\begin{pmatrix} \times & U_{l-1\ v-1} & U^*_{l-1\ v+1} \\ U^*_{l-1\ v+1} & \times & U_{l-1\ v-1} \\ U_{l+1\ v+1} & U^*_{l+1\ v-1} & \times \end{pmatrix}$$

odd

$$\begin{pmatrix} U_{l-1\ v-1} & U^*_{l-1\ v+1} & \times \\ U^*_{l-1\ v+1} & \times & U_{l-1\ v-1} \\ \times & U_{l+1\ v+1} & U^*_{l+1\ v-1} \end{pmatrix}$$

6-Plaquette



$$\Omega_l := U_{l-1\ v-1} U^*_{l-1\ v+1} U_{l-1\ v-1} U^*_{l-1\ v+1} U_{l+1\ v+1} U^*_{l+1\ v-1}$$

$$\sum_{even} \Omega = \sum_{odd} \Omega = \underline{2/9\, C - 1/3\, G + 1/9} \longleftarrow$$

purely real

More Flavour-Symmetric Constarints:

$$8C^3 - 27F^2(CG - AF) = 0 \quad \Longleftrightarrow \quad |U_{\alpha i}|^2 = 1/3$$

$$8B^3 - 27F^2(BG - DF) = 0 \quad \Longleftrightarrow \quad |U_{\alpha i}|^2 = |U_{\beta i}|^2$$

$$|K| = 0 \qquad J^2 = 0 \quad \Longleftrightarrow \quad |U_{\alpha i}|^2 = 0$$

$$J^2 = 1/108 - G/18 + 2C/27 - F^2/36$$

$$|K| = \mathrm{Det}\ K = A/27 + F^3/54 - FC/27 - F/54$$

$$|K| = 0 \qquad J^2 \to 0$$

$$\Rightarrow \varphi_{\alpha i} \approx \underline{90^o} \quad \text{!!!}$$

Completely Symmetric CKM *P*-matrix:

$$B = D \quad \Longleftrightarrow \quad |V_{\alpha i}|^2 = |V_{i\alpha}|^2$$

Extremise a "Potential", e.g.:

$$\left.\begin{array}{l} V(A,C,F,J) = A^2 + C^2 + F^2 + J^2 \\ \partial_A V = 0 \quad \partial_C V = 0 \quad \partial_F V = 0 \quad \partial_J V = 0 \end{array}\right\} \Rightarrow \quad A = C = F = J = 0$$

Tri-Bi-Maximal Mixing !!!

EXTREMISATION: A TRIVIAL EXAMPLE

In the SM:

Yukawa couplings x, y, z

$$m_e = x < \varphi >$$
$$m_\mu = y < \phi >$$
$$m_\tau = z < \phi >$$

$$< \phi > = \frac{v}{\sqrt{2}} \approx 180 \ \text{GeV}$$

Add to SM Action, the determinant : $\text{Det}\, L = m_e \, m_\mu \, m_\tau$

(taken here to be dimensionless) i. e. $Action = x \, y \, z$

$$\begin{aligned} \partial_x A &= y \, z = 0 \\ \partial_y A &= x \, z = 0 \\ \partial_z A &= x \, y = 0 \end{aligned} \quad \xrightarrow{e.g.} \quad \left. \begin{aligned} x &= 0 \\ y &= 0 \\ z &\neq 0 \end{aligned} \right\}$$

i.e. 2 zero mass
1 non-zero!

NOT BAD!!

This notion appeared in:
P.F. Harrison and W. G. Scott Phys. Lett. B 333 (1994) 471. hep-ph/9406351

“Covariant Extremisation of Flavour-Symmetric….”

P.F. Harrison, and W. G. Scott Phys. Lett. B 628 (2005) 93. hep-ph/0508012

The Jarlskog Commutator: $C = -i[L, N] \Longrightarrow \mathrm{Tr}\ C = 0$

Characteristic Equationn: $C^3 - \xcancel{(\mathrm{Tr}\, C)}\, C^2 + (\mathrm{Tr}\, C^2/2)\, C - (\mathrm{Tr}\, C^3/3) = 0$

We extremise wrt Mass Matrices theselves: $\left\{ \begin{array}{l} L = M_l \rightarrow M_l M_l^{\dagger} \\ N = M_\nu \rightarrow M_\nu M_\nu^{\dagger} \end{array} \right.$

Extremising: $\mathrm{Tr}\ C^3/3$ $(= \mathrm{Det}\ C)$

$$\partial_L\ \mathrm{Tr}\ C^3/3 = -i[N, C^2]^T = 0$$

$$\partial_N\ \mathrm{Tr}\ C^3/3 = +i[L, C^2]^T = 0$$

3 x 3 Max

$$\Longrightarrow \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

Extremising: $\mathrm{Tr}\ -C^2/2$ (=ΣPrincipal Minors C)

$$\partial_L\ \mathrm{Tr}\ -C^2/2 = +i[N, C]^T = 0$$

$$\partial_N\ \mathrm{Tr}\ -C^2/2 = -i[L, C]^T = 0$$

2 x 2 Max

$$\Longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

et perms.

Extremising: V(C)

$$V(C) = \mathrm{Tr}\ C^3/3 + r\ \mathrm{Tr}\ C^2/2 \rightsquigarrow$$

“The Simplest Neutrino Mass Matrix”
P. F. Harrison and W. G. Scott
PLB 594 (2004) 324. hep-ph/0403278

Extremise wrt the Mass matrices themselves!

Exploit Matrix Calculus Theorem

$$\partial_X \, Tr \; AX = A^T \qquad (\partial_X := \partial \; / \partial X)$$

Where A is any constant matrix and X is a variable matrix.

Apply to Extremise Tr C^3 $\quad (C := -i\,[L,N]\,)$

$$\partial_L \, Tr \;\; C^3/3 \; = \; -i[N,C^2]^T \; = 0$$
$$\partial_N \, Tr \;\; C^3/3 = \; +i[L,C^2]^T \; = 0$$

Weak-Basis
Covariant *!!*

Apply to Extremise Tr C^2

$$\partial_L \, Tr \;\; C^2/2 \; = \; -i[N,C]^T \; = 0$$
$$\partial_N \, Tr \;\; C^2/2 = \; +i[L,C]^T \; = 0$$

c.f. Maxell/Yang-Mills

$$A = Tr\; F^2/2$$
$$F = [\nabla_\mu, \nabla_\nu]$$
$$\Downarrow$$
$$[\nabla_\mu, [\nabla_\mu, \nabla_\nu]\,] = 0$$

The “Epsilon” Phase Convention*

The usual (charged-lepton) flavour basis has not been completely defined.

There remains the freedom to re-phase the fields such that he imaginary part of the neutrino mass matrix is proportional to the epsilon matrix

“the epsilom matrix”:

$$\varepsilon = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\text{i.e.} \quad \mathrm{Im}\, N = d \times \varepsilon$$

Incredible but true!!

$$N = M_\nu M_\nu^\dagger = \begin{array}{c} \\ \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \nu_e & \nu_\mu & \nu_\tau \end{array} \\ \begin{pmatrix} a & z+id & y-id \\ z-id & b & x+id \\ y+id & x-id & c \end{pmatrix} \end{array}$$

Now the 7 parameters a, b, c, d, x, y, z encode directly the 3 neutrino masses and the usual 4 mixing parameters.

*See Footnote 1 of: “The Simplest Neutrino Mass Matrix”
P. F. Harrison & W. G. Scott Phys Lett. B B594 (2004) 324. hep-ph/0403278

Try a simple linear combination of the two:

$$V(C) = Tr\,(C^3/3)\; + r\,Tr\,(-C^2/2) \qquad\qquad C = -i[L,N]$$

Take r to be a constant with dimensions of (mass)^2

With the “Magic-Square constraint” imposed there are analytical solutions:

$$x = \pm\frac{\sqrt{XYZ}}{X} \qquad X = d^2 - r\frac{d(m_\mu - m_\tau)}{(m_e - m_\mu)(m_\tau - m_e)}$$

$$y = \pm\frac{\sqrt{YZX}}{Y} \qquad Y = d^2 - r\frac{d(m_\tau - m_e)}{(m_\mu - m_\tau)(m_e - m_\mu)}$$

$$z = \pm\frac{\sqrt{ZXY}}{Z} \qquad Z = d^2 - r\frac{d(m_e - m_\mu)}{(m_\tau - m_e)(m_\mu - m_\tau)}$$

In practice:

$$r/d = 0.163\ \mathrm{GeV}^2$$
$$\rightarrow\quad h = 0.035$$

$$P = \begin{pmatrix} 0.48 & 0.33 & 0.19 \\ 0.41 & 0.33 & 0.25 \\ 0.11 & 0.33 & 0.55 \end{pmatrix}$$

✗

In general, for sufficiently extreme hierarcy $h \rightarrow 0$, we are close to the pole at $X \rightarrow 0$, i.e. $x \rightarrow \infty$ and we have $|x| >> y, z$, whereby the “Simplest” assumption must hold. ✓

In this sense this V(C) above points to the “Simplest Neutrino Mass Matrix” despite that in practice (in actuality!) the hierarchy *h* is too large!! ✓

”The Simplest Neutrino Mass Matrix”

P. F. Harrison and W. G. Scott Phys Lett. B594 (2004) 324. hep-ph/0403278.

In the charged-lepton flavour basis, ie. where M_l Is diagonal, we impose:

“Democracy Symmetry”

$$[\mathcal{D}, M_\nu] = 0$$

Ie. M_ν commutes with $\mathcal{D} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ the “democracy operator”

“Mu-Tau Reflection Symmetry” (“mutautivity”)

$$(\mathcal{E}^T M_\nu \mathcal{E})^* = M_\nu \longleftarrow$$

Note definition includes a complex conjugation

$\mathcal{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ the “μτ-exchange operator”

Finally, implementing the “Simplest” Condition:

$$\boxed{M_\nu = aI + x\mathcal{E} + d\varepsilon}$$

$$U = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \\ \hline \sqrt{\frac{2}{3}}c_\chi & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}s_\chi \\ -\frac{c_\chi}{\sqrt{6}} + i\frac{s_\chi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{s_\chi}{\sqrt{6}} - i\frac{c_\chi}{\sqrt{2}} \\ -\frac{c_\chi}{\sqrt{6}} - i\frac{s_\chi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{s_\chi}{\sqrt{6}} + i\frac{c_\chi}{\sqrt{2}} \end{array}$$

$$\sin\theta_{13} = \sqrt{2/3}\,\sin\chi = \sqrt{2\Delta m^2_{sol} / 3\Delta m^2_{atm}} \approx \underline{0.13 \pm 0.03}$$

CONCLUSIONS

1) "Tri-BiMaximal Mixing" has useful partners "Tri-χ-Maximal Mixing", and "Tri-φ-Maximal Mixing" and more generally "Tri- χφ-Maximal Mixing" (now "ν2-Trimaximal Mixing") which are also consistent with the data.

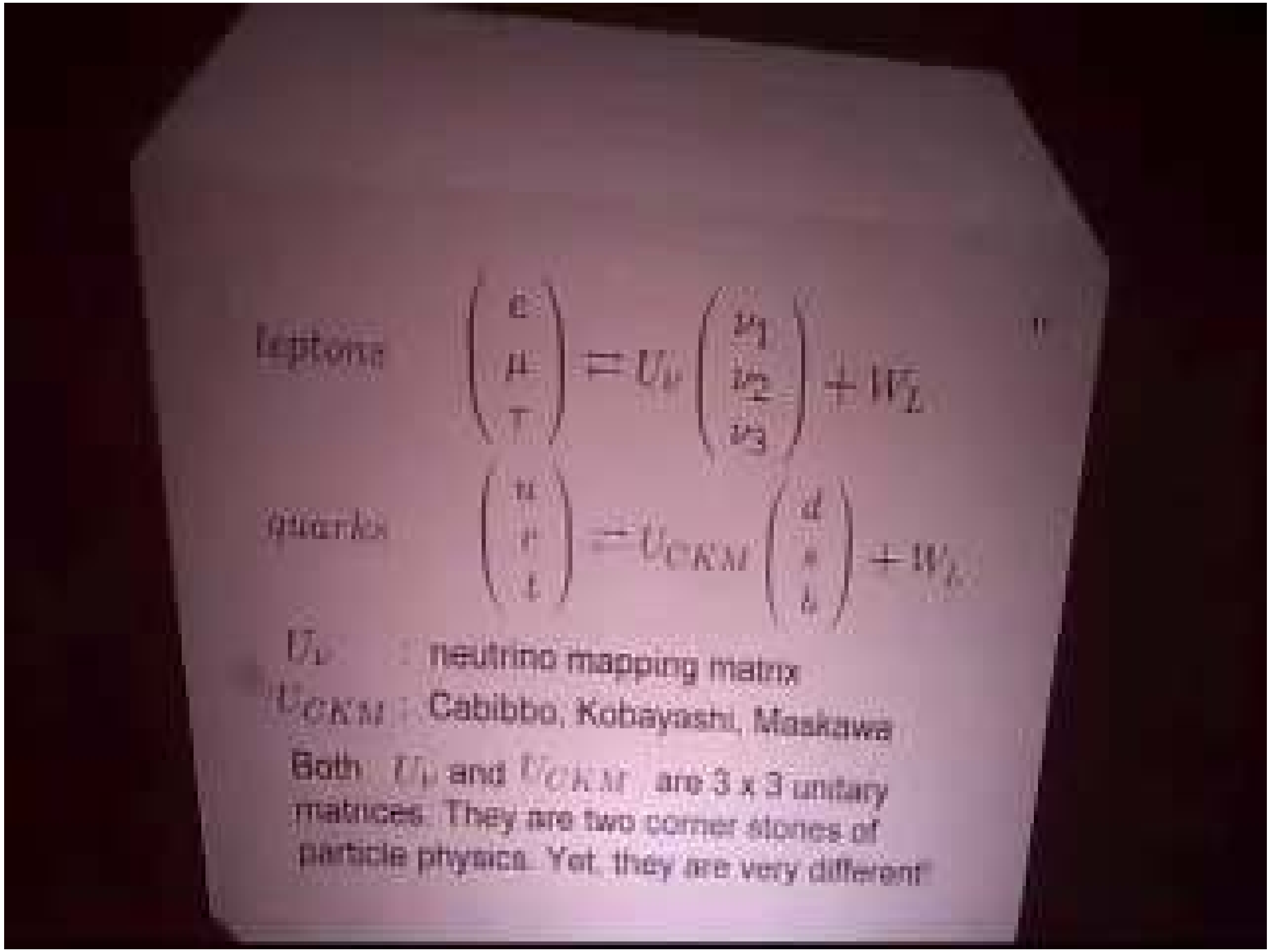
2) We have introduced 6 New Flavour-Symmetric Mixing Observables, A,B,C,D,F,G which like the Jarlskogian J can be used to constrain the mixings in an entirely flavour-symmetric way.

3) A programme of Extremisiing Flavour Symmetric Jarlskog Invariants, Is under way with the aim of constraining both Mixings and Masses. Thus far the best that can be said is that our results point towards "The Simplest…" PLB 594 (2004) 324 (hep-ph/0403278) and $\Theta_{13} \sim 0.13$.

Again T. D. Lee's lecture (a 2nd clip- from earlier in his talk) Inspirational for anyone working on fermion mixing and flavour etc. :

"….these two 3 x 3 matrices (CKM and MNS) are the cornerstones of particle physics… ….but do we understand them???"

T. D. Lee CERN colloquium Aug 2007



leptons $$\begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} = U_\nu \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} + W_L$$

quarks $$\begin{pmatrix} u \\ c \\ t \end{pmatrix} = U_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix} + W_L$$

U_ν : neutrino mapping matrix

U_{CKM} : Cabibbo, Kobayashi, Maskawa

Both U_ν and U_{CKM} are 3 x 3 unitary matrices. They are two corner stones of particle physics. Yet, they are very different!