Finite size effects on kinks and solitons

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Motivation and Outline

- Quantum fields on compact spaces

- Volume dependence of the spectrum (Lüscher corrections)

- Semiclassical quantization of twisted ϕ^4 kink on S^1

- Zeta function regularization without complete knowledge of fluctuation spectrum

- Comparing analytic semiclassical results for sine-Gordon soliton on S^1 with numerical results

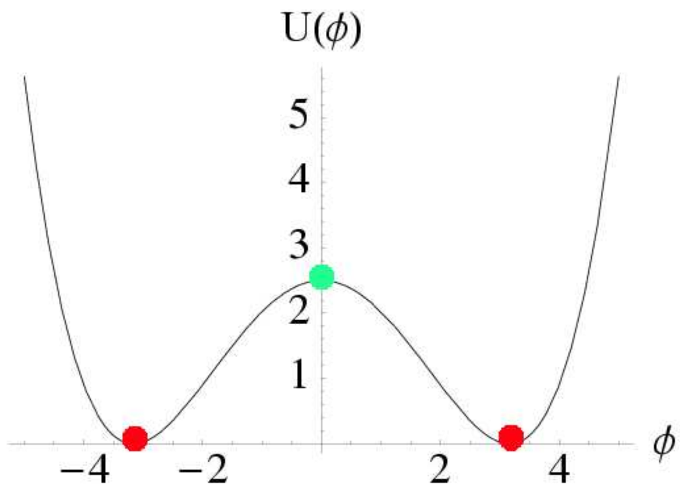
Twisted ϕ^4 kink (Sakamoto et al. 1999, Mussardo et al. 2006)

- Scalar field on S^1 with circumference R

$$S = \int d^2x \left[\partial_\mu \phi \partial^\mu \phi - U(\phi)\right], \qquad U(\phi) = \frac{\lambda}{4}(\phi^2 - \frac{m^2}{\lambda})^2$$

- Kinks as static antiperiodic solutions of

$$\frac{d^2\phi}{dx^2} - U'(\phi) = 0, \qquad \phi(x+R) = -\phi(x)$$



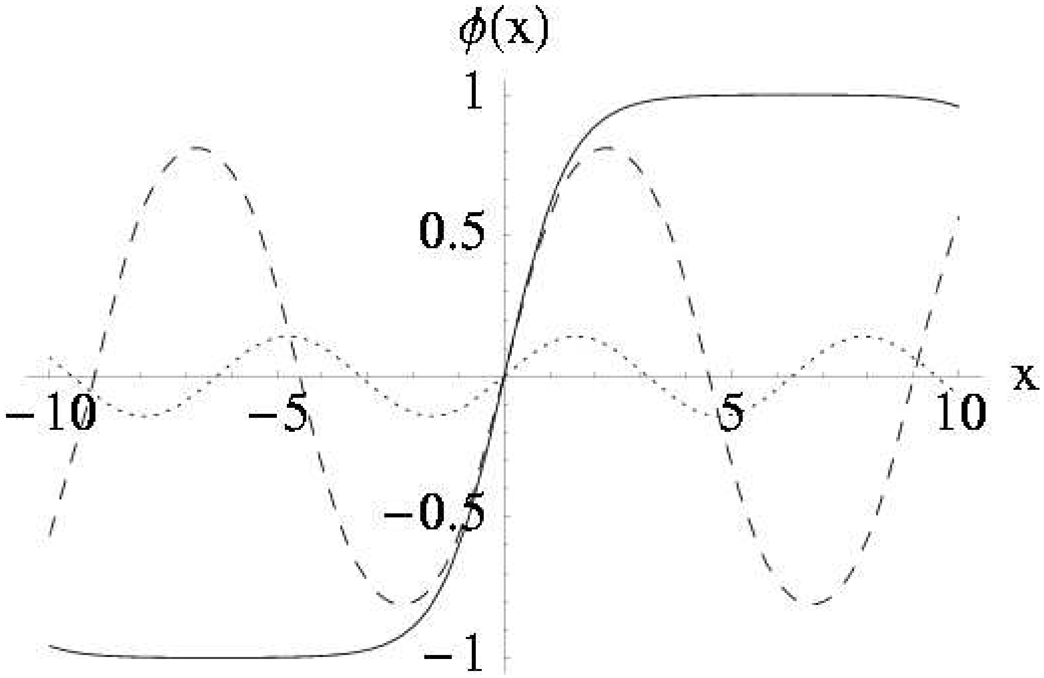
Two sectors

- $R < R_0 = \frac{\pi}{m}$: Constant field configuration: $\phi_0(x) = 0$ with energy $E_{\phi=0}(R) = \frac{m^4}{4\lambda} R$
- $R > R_0$: Twisted kink solution:

$$\phi_k(x) = \frac{m}{\sqrt{\lambda}} \sqrt{\frac{2k^2}{k^2+1}} \mathrm{sn}\left(\frac{mx}{\sqrt{k^2+1}}, k \right), \qquad R = \frac{2}{m} \sqrt{1+k^2} \mathbf{K}(k)$$

with energy

$$E_{cl}(k) = \frac{m^3}{6\lambda} \frac{1}{(k^2+1)^{\frac{3}{2}}} \left[(k^2-1)(5+3k^2)\mathbf{K}(k) + 8(k^2+1)\mathbf{E}(k) \right]$$



For $R \to \infty$ ordinary ϕ^4 kink is recovered:

$$\phi_k(x) \stackrel{k\to 1}{\to} \frac{m}{\sqrt{\lambda}} \tanh(mx/\sqrt{2}), \qquad E_{cl} \stackrel{k\to 1}{\to} \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}$$

1-Loop fluctuations for twisted ϕ^4 model

- Semiclassical expansion about classical solution $\phi_0(x)$

$$\phi(x,t) = \phi_0(x) + e^{i\sqrt{\lambda}t}\chi(x)$$

yields fluctuation equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U''(\phi_0(x))\right]\chi_n(x) = \lambda_n \chi_n(x)$$

- Fluctuation spectrum for $R < R_0$

$$\lambda_n = \frac{(2n+1)^2}{R^2}\pi^2 - m^2$$

$\Rightarrow$ For $R > R_0$ eigenvalues λ_n become negative
$\Rightarrow$ Constant field configuration is instable for $R > R_0$

1-Loop fluctuations for $R > R_0$

- For $R > R_0$ the fluctuation equation is the $n = 2$ Lamé equation:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\bar{x}^2} + n(n+1)k^2\mathrm{sn}^2(\bar{x},k)\right]\chi(\bar{x}) = h\chi(\bar{x})$$

where $h = \left(\frac{\lambda}{m^2}+1\right)(1+k^2)$ and $\bar{x} = mx/\sqrt{1+k^2}$

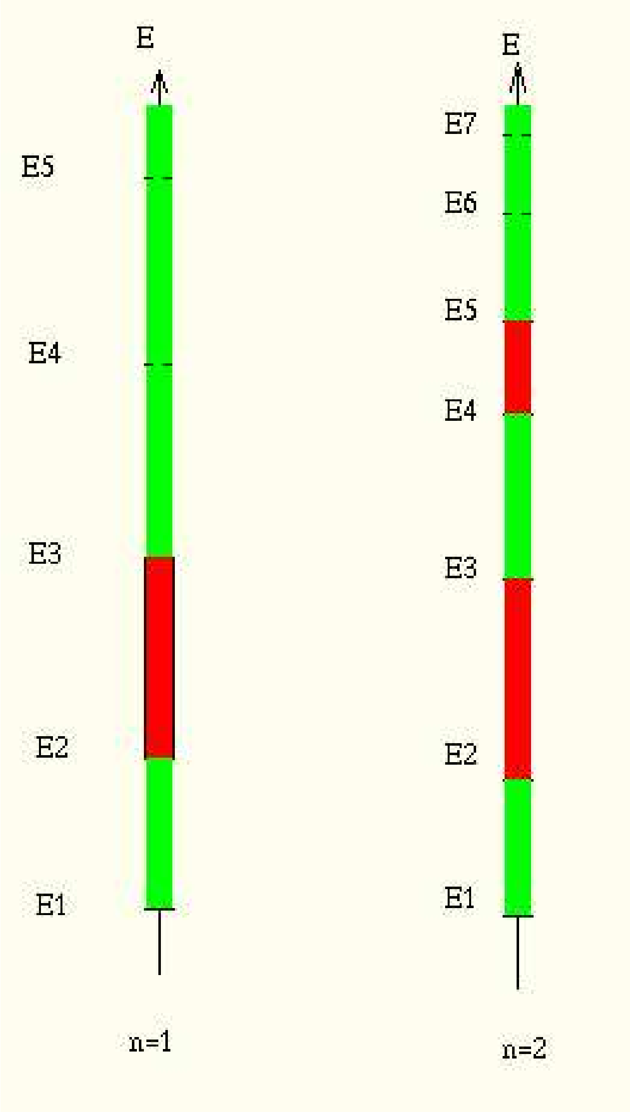
- Quasi-exactly solvable differential equation

$\Rightarrow$ only five eigenvalues are analytically known:

$$\bar{\lambda}_1 = 1 - 2\frac{\sqrt{1-k^2(1-k^2)}}{k^2+1}, \qquad \bar{\lambda}_2 = 0, \qquad \bar{\lambda}_3 = \frac{3k^2}{1+k^2},$$
$$\bar{\lambda}_4 = \frac{3}{1+k^2}, \qquad \bar{\lambda}_5 = 1 + 2\frac{\sqrt{1-k^2(1-k^2)}}{k^2+1}$$

- finite-gap property

Finite gap spectrum of Lamé equation



Semiclassical energy

$$E = E_{cl} + \underbrace{\sum_{n=0}^{\infty} \sqrt{\lambda_n}}_{E_{1-loop}}$$

- For regularization of divergent expression define the spectral zeta function

$$\zeta_D(s) = \mu^{1+2s} \sum_{n=0}^{\infty} \lambda_n^{-s}, \qquad \mathrm{Re}(s) > s_0 > -1/2$$

$$\Longrightarrow E_{1-loop} = \frac{1}{2}\zeta_D(-1/2)$$

where $\zeta_D(-1/2)$ is understood as the analytic continuation

Integral representation of Zeta functions (Kirsten 2000, Bordag 2001)

Spectral discriminant $\Delta(\lambda) = 0 \quad \Leftrightarrow \quad \lambda$ eigenvalue of $D f_\lambda(x) = \lambda f_\lambda(x)$

$$\zeta_D(s) = \frac{1}{2\pi i}\mu^{1+2s} \int_\gamma d\lambda \lambda^{-s} \underbrace{\frac{\partial}{\partial\lambda} \ln \Delta(\lambda)}_{R(\lambda)} \stackrel{\text{Residue}}{=} \mu^{1+2s} \sum_{n=0}^{\infty} \lambda_n^{-s}$$

- Wrapping the contour along the cut for suitable values of s:

$$\zeta_D(s) = -\frac{\sin(\pi s)}{\pi}\mu^{1+2s} \int_0^\infty d\lambda \lambda^{-s} R(-\lambda)$$

- Connection of spectral discriminant $\Delta(\lambda)$ and quasi (Bloch) momentum $p(\lambda)$

$$\Delta(\lambda) = 2\cos(Rp(\lambda)) \pm 2$$

with

$$f_\lambda(x+R) = e^{iRp(\lambda)} f_\lambda(x)$$

Bethe ansatz equations for $n = 2$ Lamé equation

- The general solution for

$$-\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 6k^2 \mathrm{sn}^2(x,k) f(x) = \lambda f(x)$$

is given by (Whittaker)

$$f(x) = \frac{H(x+\alpha_1)H(x+\alpha_2)}{\Theta(x)^2} e^{-x(Z(\alpha_1)+Z(\alpha_2))}$$

with quasi-momentum

$$p(\alpha_1,\alpha_2) = iZ(\alpha_1) + iZ(\alpha_2)$$

if the additional parameters α_1, α_2 fulfil the following two transcendental equations (Bethe ansatz equation)

$$\mathrm{sn}(\alpha_1)\mathrm{cn}(\alpha_1)\mathrm{dn}(\alpha_1) + \mathrm{sn}(\alpha_2)\mathrm{cn}(\alpha_2)\mathrm{dn}(\alpha_2) = 0$$
$$(\mathrm{cn}(\alpha_1)\mathrm{ds}(\alpha_1) + \mathrm{cn}(\alpha_2)\mathrm{ds}(\alpha_2))^2 - \mathrm{ns}^2(\alpha_1) - \mathrm{ns}^2(\alpha_2) = -\lambda$$

Construction of Spectral discriminant (Pawellek arXiv:0802.0710 (hep-th))

- Bethe ansatz equations

$$\mathrm{sn}(\alpha_1)\mathrm{cn}(\alpha_1)\mathrm{dn}(\alpha_1) + \mathrm{sn}(\alpha_2)\mathrm{cn}(\alpha_2)\mathrm{dn}(\alpha_2) = 0$$
$$(\mathrm{cn}(\alpha_1)\mathrm{ds}(\alpha_1) + \mathrm{cn}(\alpha_2)\mathrm{ds}(\alpha_2))^2 - \mathrm{ns}^2(\alpha_1) - \mathrm{ns}^2(\alpha_2) = -\lambda$$

- Solutions of Bethe ansatz equations

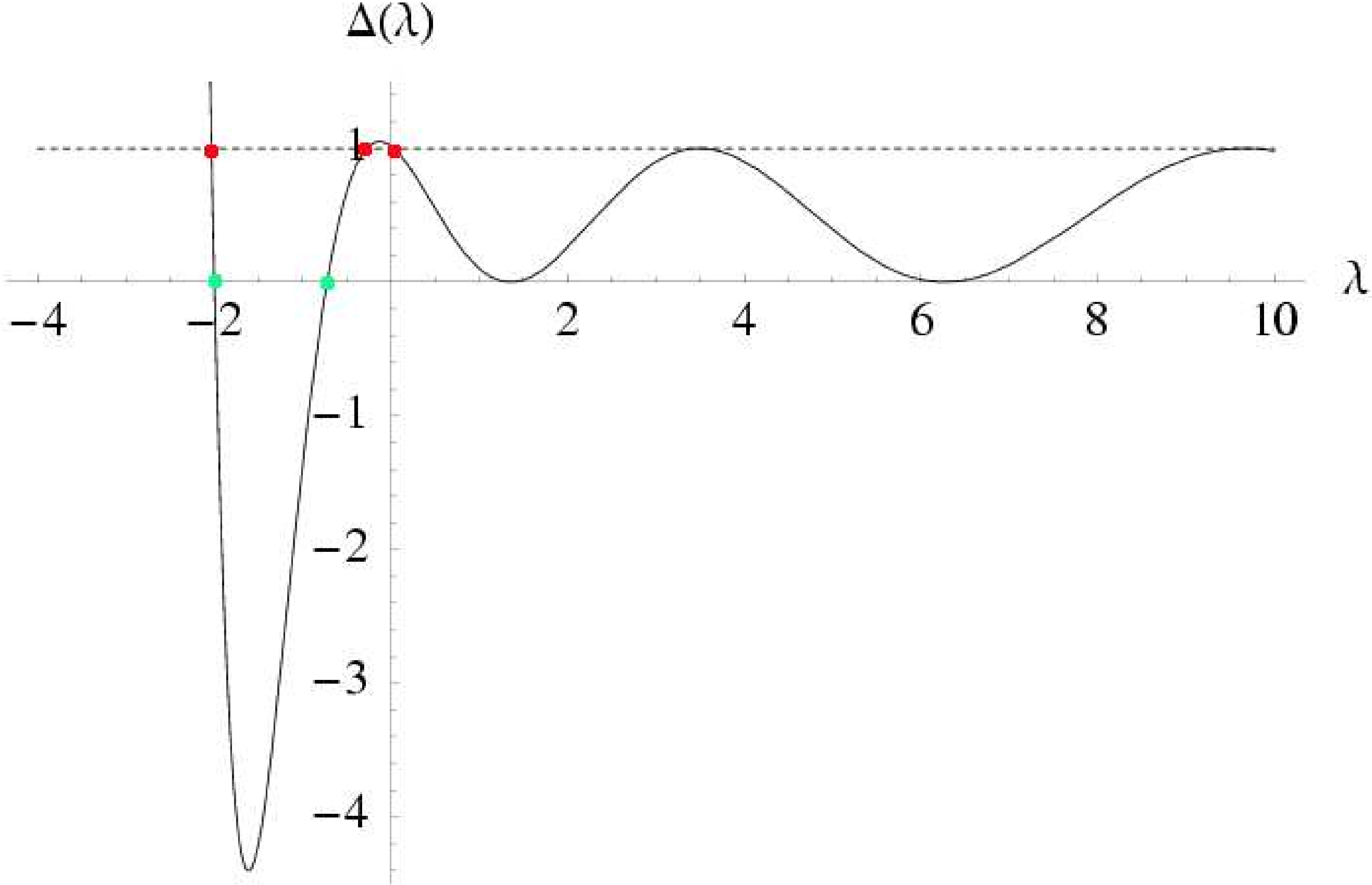
$$\mathrm{sn}^2\alpha_{1,2} = \frac{4(1+k^2)-\lambda}{6k^2} \pm \frac{1}{2k^2}\sqrt{g_2(k) - \frac{1}{3}(\lambda - 2(1+k^2))^2}$$

with

$$g_2(k) = \frac{4}{3}(1 - k^2(1-k^2))$$

- Spectral discriminant for $n = 2$ Lamé equation

$$\Delta(\lambda) = \cos^2\left(\mathbf{K}(k)p(\lambda)\right)$$



Δ(λ)
λ
−4
−2
2
4
6
8
10
1
−1
−2
−3
−4

Renormalized 1-loop energy for the twisted kink

- Renormalization condition for $R > R_0$

$$E_{ren} \to 0 \text{ for } \ m \to \infty$$

$$E_{ren}(k) = \frac{1}{2\pi} \int_{\sqrt{2}m}^{\infty} d\kappa \sqrt{\kappa^2 - 2m^2} \left[\mathcal{R}(\kappa) - r_0 - \frac{r_1}{\kappa^2} \right]$$

with and

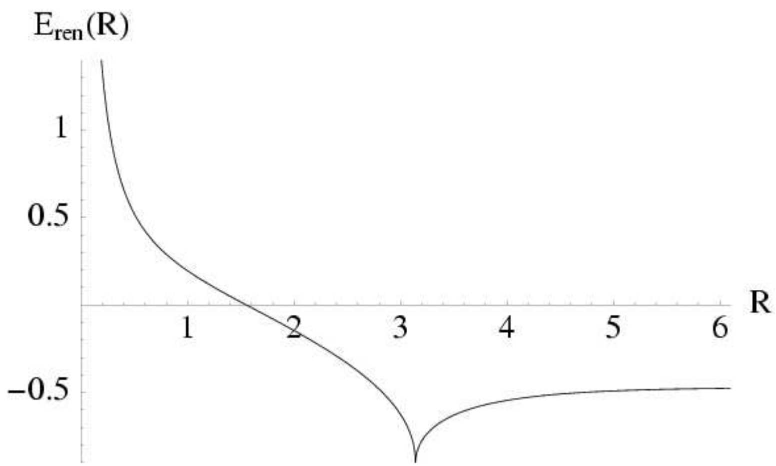
$$r_0 = -R = -\frac{2}{m}\sqrt{1+k^2}\mathbf{K}(k), \qquad r_1 = -\frac{3m}{\sqrt{k^2+1}}\left[(k^2-1)\mathbf{K}(k) + 2\mathbf{E}(k)\right]$$

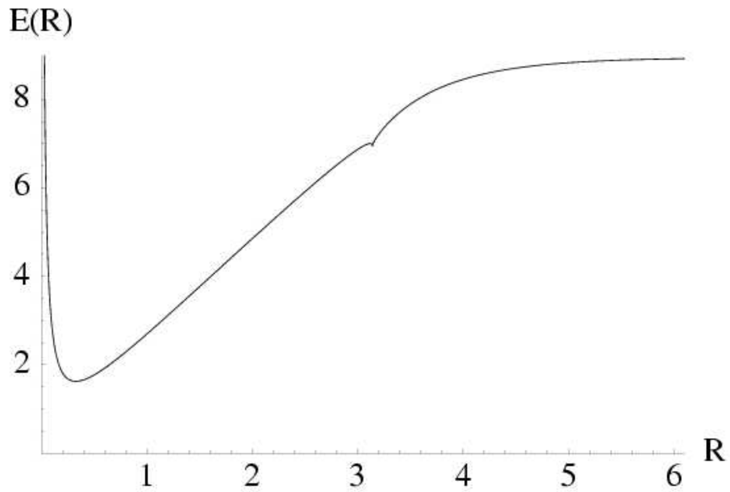
- Renormalization condition for $R < R_0$

$$E_{ren,R<R_0}(R_0) = E_{ren}(k=0)$$

$$\Rightarrow E_{ren}(R) = -\frac{R}{2\pi} \int_{\sqrt{2}m}^{\infty} d\kappa \sqrt{\kappa^2 - 2m^2} \left[\frac{\kappa \tanh\left(\frac{R}{2}\sqrt{\kappa^2 - 3m^2}\right)}{\sqrt{\kappa^2 - 3m^2}} - 1 - \frac{3m^2}{2\kappa^2} \right]$$

Results





- For $R < R_0$ a local minimum R_{min} of total energy $E = E_{cl} + E_{ren}$ exist

$$R_{min} \approx \frac{1}{m}\sqrt{\frac{\lambda}{m^2}} \; (= 0.31 \text{ for } m = 1, \lambda = 0.1)$$

- For $k \to 1$ or $R \to \infty$ the value of the 1-loop energy approaches the well known result of the standard kink:

$$E_{ren} \to \left(\frac{1}{2\sqrt{6}} - \frac{3}{\sqrt{2}\pi}\right) m = -0.4711m, \qquad k \to 1$$

Sine-Gordon on S^1

$$U(\phi) = \frac{m^2}{\beta^2}(1 - \cos(\beta\phi))$$

- periodic b.c.: $\phi(x+R) = \phi(x) + \frac{2\pi}{\beta}$

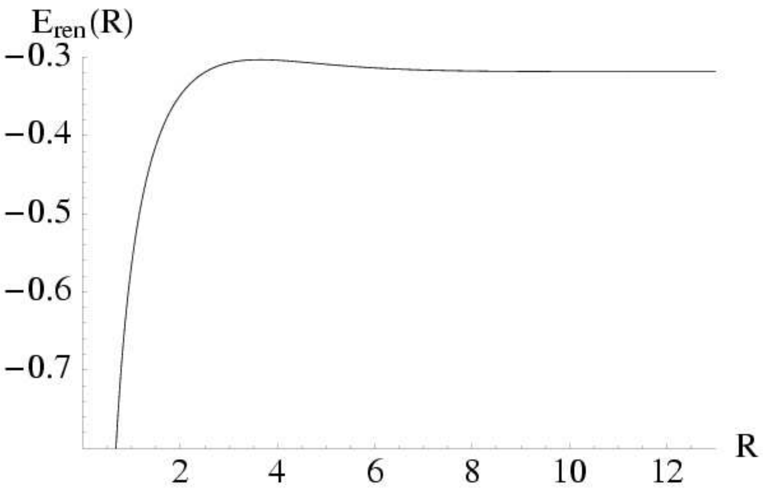
$$\phi_k(x) = \frac{\pi}{\beta} + \frac{2}{\beta}\mathrm{am}\left(\frac{m(x-x_0)}{k}, k\right), \; R = \frac{2k}{m}\mathbf{K}(k)$$

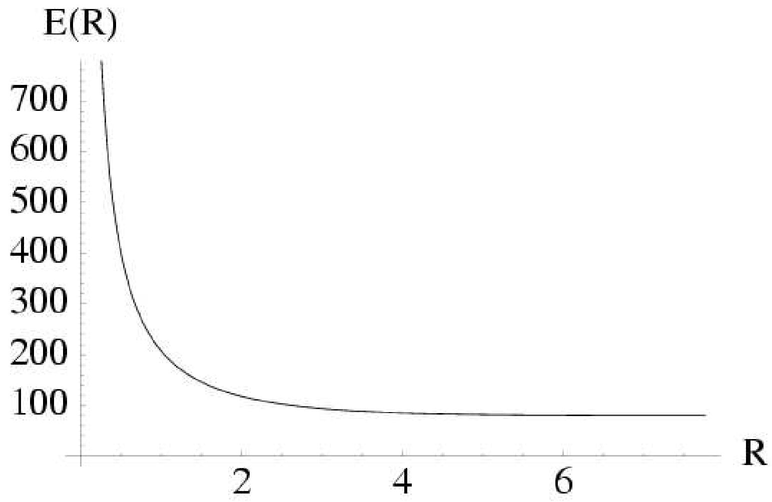
- anti-periodic b.c.: $\phi(x+R) = -\phi(x) + \frac{2\pi}{\beta}$

$$\phi_k(x) = \frac{2}{\beta}\arccos\left(k\,\mathrm{sn}(m(x-x_0))\right), \; R = \frac{2}{m}\mathbf{K}(k)$$

- In both cases the fluctuation equation is the $n = 1$ Lame equation

Results for periodic soliton





- For $R \to 0$ E_{ren} behaves as the Casimir energy of a periodic massless field:

$$E_{ren}(R) \to -\frac{\pi}{6R}$$

- For $R \to \infty$ one gets the mass correction of the Sine Gordon soliton:

$$E_{ren}(R) \to -\frac{m}{\pi}$$

Comparison with exact numerical data

- Using integrability of Sine-Gordon model leads to Non-linear-integral equations (NLIE)(Feverati et.al 1999)
- Compare semiclassical with numerical results for $\beta^2 = 5.585$

l	semiclassical	NLIE	relative deviation
0.5	6.07727	6.080571	0.0005
1	3.12108	3.126706	0.0018
1.5	2.17017	2.177411	0.0033
2	1.71874	1.727224	0.0049
2.5	1.46556	1.475004	0.0064
3	1.31020	1.320353	0.0077
4	1.14250	1.153188	0.0093
5	1.06557	1.075376	0.0091

$\Longrightarrow$ Good agreement with semiclassical results outside of the expected regime of validity $\beta^2 \ll 1$

Summary and Outlook

- Analytic solution for Bethe ansatz equations for $n = 2$ Lamé equation
  $\Rightarrow$ Analytic result for semiclassical energy of twisted ϕ^4 kink on S^1 for all R
  $\Rightarrow$ Only the last integral is calculated numerically

- Validity of semiclassical results for Sine-Gordon solitons on S^1 outside the expected regime

- SUSY sine-Gordon solitons on S^1?

- Finite-gap flutuations in more complex theories? Relevant for Semiclassical quantization of strings?

Solution for quasi-momentum and spectral discriminant

- Quasi-momentum

$$p(\lambda) = iZ\left[\mathrm{sn}^{-1}\left[\sqrt{\frac{4(1+k^2)-\lambda}{6k^2}+\frac{1}{2k^2}\sqrt{g_2(k)-\frac{1}{3}(\lambda-2(1+k^2))^2}}\right],k\right]+ +iZ\left[-\mathrm{sn}^{-1}\left[\sqrt{\frac{4(1+k^2)-\lambda}{6k^2}-\frac{1}{2k^2}\sqrt{g_2(k)-\frac{1}{3}(\lambda-2(1+k^2))^2}}\right],k\right]$$

- Spectral discriminant for the antiperiodic eigenfunctions

$$\Delta(\lambda) = 4\cos^2\left(\mathbf{K}(k)p(\lambda)\right)$$

- The resolvent can then be written as

$$R(\lambda) = -\mathbf{K}(k)\tan[\mathbf{K}(k)p(\lambda)]p'(\lambda)$$

Derivative of the quasi-momentum

$$p'(\lambda) = -\frac{i}{2}\frac{(\lambda-\mu_1)(\lambda-\mu_2)}{\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)(\lambda-\lambda_5)}}$$

with

$$\mu_{1,2} = \frac{3}{2}\frac{\mathbf{E}(k)}{\mathbf{K}(k)} + \frac{5}{2}k^2 + 1 \pm \frac{3}{2}\sqrt{\frac{2}{3}g_2(k) + \left(\frac{\mathbf{E}(k)}{\mathbf{K}(k)} - \frac{(2-k^2)}{3}\right)^2}$$

$$\lambda_1 = 1+k^2, \quad \lambda_2 = 1+4k^2, \quad \lambda_3 = 4+k^2$$

$$\lambda_{4,5} = 2(1+k^2) \pm 2\sqrt{1-k^2(1-k^2)}$$

- λ_i are the eigenvalues and μ_i are the first two local extrema of $\Delta(\lambda)$,
- $p(\lambda)$ and $p'(\lambda)$ live on $g=2$ Riemann surface $y^2 = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)(\lambda-\lambda_5)$

The parameters are given by

$$\mu_{1,2} = \frac{m^2}{2(1+k^2)}\left(3\frac{\mathbf{E}(k)}{\mathbf{K}(k)} - (4+k^2) \pm 3\sqrt{\frac{2}{3}g_2(k) + \left(\frac{\mathbf{E}(k)}{\mathbf{K}(k)} - \frac{2-k^2}{3}\right)^2}\right),$$

$$\kappa_1^2 = -2m^2, \quad \kappa_2^2 = \frac{k^2-2}{1+k^2}m^2, \quad \kappa_3^2 = \frac{1-2k^2}{1+k^2}m^2,$$

$$\kappa_{4,5}^2 = \left(-1 \pm \frac{2}{1+k^2}\sqrt{1-k^2(1-k^2)}\right)m^2$$