

**DOING PHYSICS WITH NON-DIAGONALIZABLE HAMILTONIANS  
AND THE SOLUTION TO THE GHOST PROBLEM  
IN FOURTH-ORDER DERIVATIVE THEORIES**

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## THINGS WE TAKE FOR GRANTED IN QUANTUM MECHANICS.....

- (1) The Hamiltonian must be Hermitian
- (2) The momentum operator must be Hermitian
- (3) In the  $[x, p] = i\hbar$  commutator the momentum operator can always be represented by  $p = -i\hbar \frac{\partial}{\partial x}$ .
- (4) States such as energy eigenstates must form a complete set
- (5) To be complete states must be normalizable
- (6) The scalar product must be given as  $\langle m|n\rangle = \delta_{m,n}$
- (7) The completeness relation must be given by  $\sum |n\rangle\langle n| = 1$
- (8) Theories in which  $\langle n|n\rangle$  is negative are unphysical and cannot be formulated in Hilbert space
- (9) The Hamiltonian must be diagonalizable

.....AIN'T NECESSARILY SO

**AND FOR THEORIES BASED ON FOURTH-ORDER DERIVATIVES....**

**ALL THESE THINGS ARE NECESSARILY NOT SO,.....**

**AND CAN ENABLE FOURTH-ORDER DERIVATIVE CONFORMAL GRAVITY TO BE A CONSISTENT THEORY OF QUANTUM GRAVITY IN FOUR SPACETIME DIMENSIONS**

# EVEN IF NON-NORMALIZABLE, STATES CAN STILL BE COMPLETE

## Completeness of non-normalizable modes

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$$\int_{-\infty}^{\infty} dw e^{-2A(w)} f_m(w) f_{m'}(w) = \delta_{m,m'} \quad (1)$$

$$\psi(w) = \sum_m a_m f_m(w) \quad (2)$$

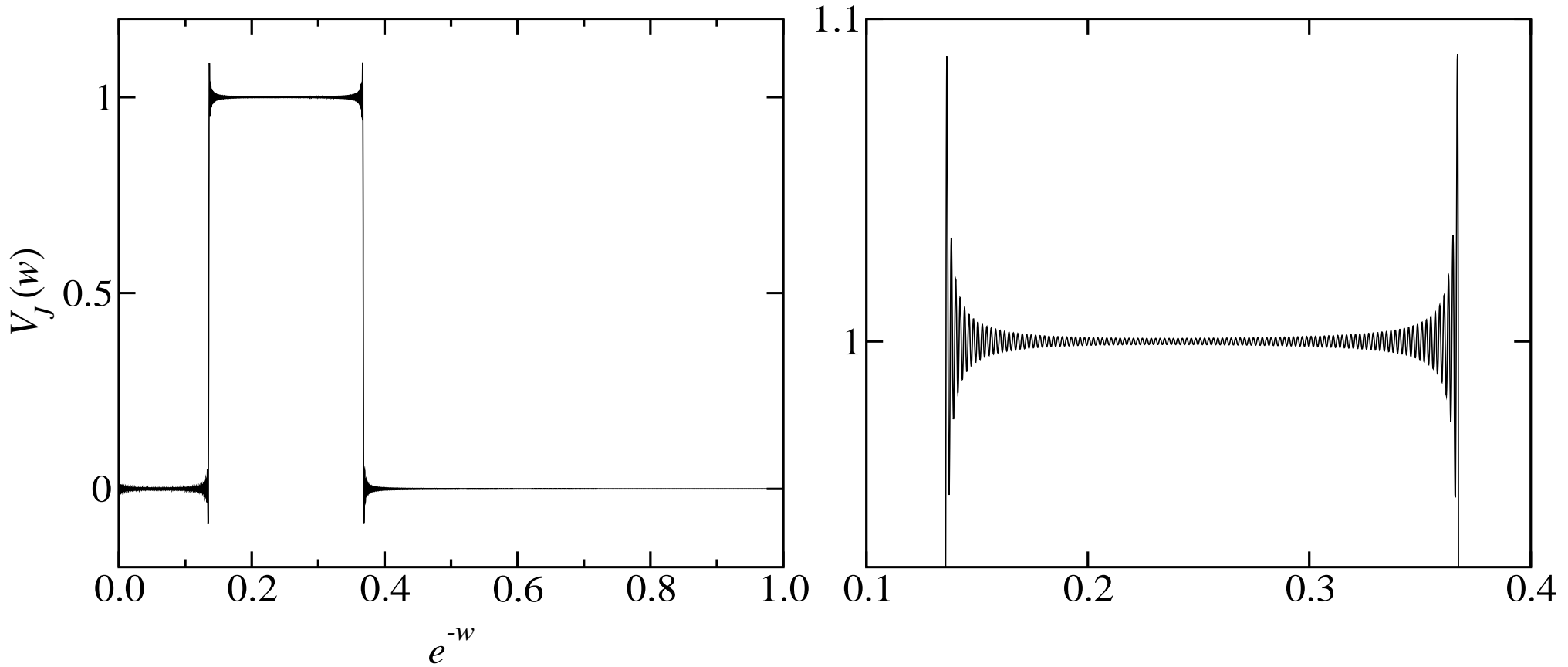
$$a_m = \int_{-\infty}^{\infty} dw e^{-2A(w)} f_m(w) \psi(w) \quad (3)$$

$$\psi(w) = \sum_m \int_{-\infty}^{\infty} dw' e^{-2A(w')} f_m(w') f_m(w) \psi(w') \quad (4)$$

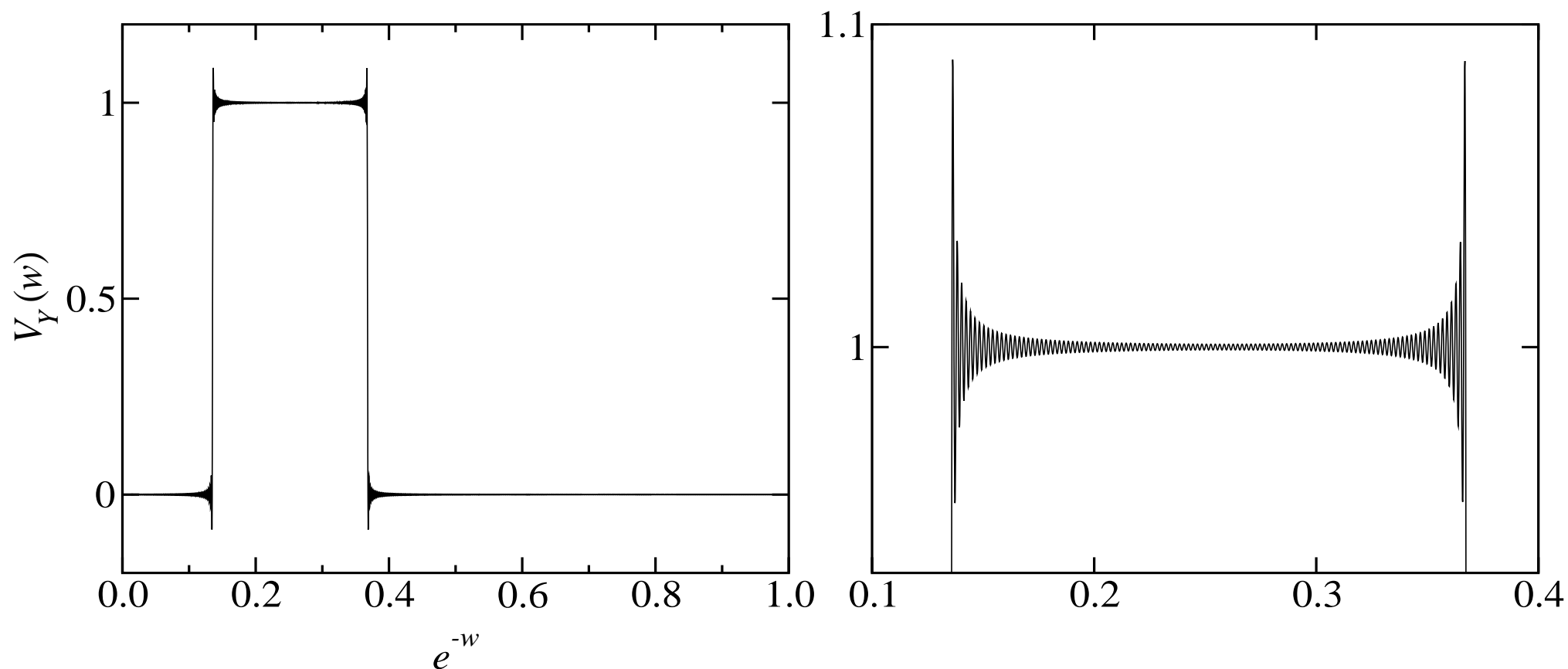
$$\psi(w) = \int_{-\infty}^{\infty} dw' \delta(w - w') \psi(w') \quad (5)$$

$$\sum_m f_m(w') f_m(w) = e^{2A(w)} \delta(w - w') \quad . \quad (6)$$

**Reconstruction of the square step  $V_J(|w|) = 1, 1 < |w| < 2, V_J = 0$  otherwise via sum  $V_J(|w|) = \sum a_i J_2(j_i e^{-|w|})$  on CONVERGENT NORMALIZABLE modes with basis states which obey  $J_1(j_i) = 0$ .**



**Reconstruction of the square step  $V_Y(|w|) = 1, 1 < |w| < 2, V_Y = 0$  otherwise via sum  $V_Y(|w|) = \sum a_i Y_2(y_i e^{-|w|})$  on DIVERGENT NON-NORMALIZABLE modes with basis states which obey  $Y_1(y_i) = 0$ .**



**CONCLUSION:** All that matters is **LINEAR** relation:  $\psi(w) = \sum_m a_m f_m(w)$ .

No need to require **BILINEAR** relation  $\sum_m f_m(w') f_m(w) = e^{2A(w)} \delta(w - w')$ .

**IMPLICATION:**  $H|\psi\rangle = E|\psi\rangle$  is linear. No reference to  $\langle\psi|\psi\rangle$ . Thus left eigenvector which obeys  $\langle L|H = \langle L|E$  need not be conjugate of right eigenvector which obeys  $H|R\rangle = E|R\rangle$ .

## GHOST PROBLEM AND UNITARITY

1. C. M. Bender and P. D. Mannheim, *No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model*, June 2007 (0706.0207 [hep-th]). Phys. Rev. Lett. **100**, 110402 (2008).
2. P. D. Mannheim, *Conformal Gravity Challenges String Theory*, Pascos-07, July 2007 (0707.2283 [hep-th]).
3. C. M. Bender and P. D. Mannheim, *Giving up the ghost*, October 2007 (0807.2607 [hep-th]). Jour. Phys. A **41**, 304018 (2008)
4. C. M. Bender and P. D. Mannheim, *Exactly solvable PT-symmetric Hamiltonian having no Hermitian counterpart*, April 2008 (0804.4190 [hep-th]). Phys. Rev. D **78**, 025022 (2008).

## PT QUANTUM MECHANICS

5. C. M. Bender, *Making Sense of Non-Hermitian Hamiltonians*, March 2007 (hep-th/070309). Rep. Prog. Phys. **70**, 947 (2007).

## PAIS-UHLENBECK FOURTH ORDER OSCILLATOR

6. P. D. Mannheim and A. Davidson, *Fourth order theories without ghosts*, January 2000 (hep-th/0001115).
7. P. D. Mannheim and A. Davidson, *Dirac quantization of the Pais-Uhlenbeck fourth order oscillator*, August 2004 (hep-th/0408104). Phys. Rev. A **71**, 042110 (2005).
8. P. D. Mannheim, *Solution to the ghost problem in fourth order derivative theories*, August 2006 (hep-th/0608154). Found. Phys. **37**, 532 (2007).

## CONFORMAL GRAVITY AND THE COSMOLOGICAL CONSTANT PROBLEM

9. P. D. Mannheim, *Alternatives to dark matter and dark energy*, May 2005 (astro-ph/0505266). Progress in Particle and Nuclear Physics **56**, 340 (2006).
10. P. D. Mannheim, *Dynamical symmetry breaking and the cosmological constant problem*, September 2008 (0809.1200 [hep-th]). Proceedings of ICHEP08.

# THE QUANTUM GRAVITY UNITARITY PROBLEM

In four spacetime dimensions invariance under

$$g_{\mu\nu}(x) \rightarrow e^{2\alpha(x)} g_{\mu\nu}(x) \quad (7)$$

leads to a unique gravitational action

$$I_W = -\alpha_g \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha_g \int d^4x (-g)^{1/2} \left[ R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} (R^\alpha{}_\alpha)^2 \right] \quad (8)$$

where  $\alpha_g$  is dimensionless and

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} + \frac{1}{6} R^\alpha{}_\alpha [g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}] - \frac{1}{2} [g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu}] \quad (9)$$

is the conformal Weyl tensor. The associated gravitational equations of motion are the **fourth-order derivative**:

$$4\alpha_g [2C^{\mu\lambda\nu\kappa}{}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}] = T^{\mu\nu} \quad (10)$$

Conformal gravity is thus a renormalizable theory of gravity since  $\alpha_g$  is dimensionless. Moreover, conformal gravity controls the cosmological constant. Specifically, in a Robertson-Walker cosmology we have  $C^{\mu\lambda\nu\kappa} = 0$ , to yield

$$T^{\mu\nu} = 0, \quad (11)$$

so unlike the double-well Higgs potential, conformal gravity knows where the zero of energy is. However, since the field equations are fourth-order derivative equations, the theory is thought to have negative norm ghost states and not be unitary.

To illustrate the issues involved, consider the typical second- plus fourth-order derivative theory:

$$I = \frac{1}{2} \int d^4x [\partial_\mu \partial_\nu \phi \partial^\mu \partial^\nu \phi - M^2 \partial_\mu \phi \partial^\mu \phi] \quad (12)$$

$$(\partial_t^2 - \nabla^2)(\partial_t^2 - \nabla^2 + M^2)\phi(\bar{x}, t) = 0 \quad (13)$$

$$D^{(4)}(k^2) = \frac{1}{k^2(k^2 - M^2)} = \frac{1}{M^2} \left( \frac{1}{k^2 - M^2} - \frac{1}{k^2} \right) \quad (14)$$

Does the relative minus sign in propagator mean ghost states with negative norm and loss of unitarity, since anticipate that one can write the propagator as

$$D(\bar{x}, \bar{x}', E) = \sum \frac{\psi_n(\bar{x})\psi_n^*(\bar{x}')}{E - E_n} - \sum \frac{\psi_m(\bar{x})\psi_m^*(\bar{x}')}{E - E_m}, \quad (15)$$

and the completeness relation as

$$\sum |n\rangle\langle n| - \sum |m\rangle\langle m| = 1. \quad (16)$$

# PAIS-UHLENBECK OSCILLATOR

$$\phi(\bar{x}, t) \sim z(t)e^{i\bar{k}\cdot\bar{x}}, \quad \omega_1 = (\bar{k}^2 + M^2)^{1/2}, \quad \omega_2 = |\bar{k}| \quad (17)$$

$$\frac{d^4 z}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2 z}{dt^2} + \omega_1^2\omega_2^2 z = 0 \quad (18)$$

$$I_{\text{PU}} = \frac{\gamma}{2} \int dt [\dot{z}^2 - (\omega_1^2 + \omega_2^2)z^2 + \omega_1^2\omega_2^2 z^2] \quad (19)$$

$$H_{\text{PU}} = \frac{p_x^2}{2\gamma} + p_z x + \frac{\gamma}{2} (\omega_1^2 + \omega_2^2) x^2 - \frac{\gamma}{2} \omega_1^2 \omega_2^2 z^2, \quad x = \dot{z} \quad (20)$$

$$[x, p_x] = i, \quad [z, p_z] = i \quad (21)$$

$$\begin{aligned} z &= a_1 + a_1^\dagger + a_2 + a_2^\dagger, \\ p_z &= i\gamma\omega_1\omega_2^2(a_1 - a_1^\dagger) + i\gamma\omega_1^2\omega_2(a_2 - a_2^\dagger), \\ x &= -i\omega_1(a_1 - a_1^\dagger) - i\omega_2(a_2 - a_2^\dagger), \\ p_x &= -\gamma\omega_1^2(a_1 + a_1^\dagger) - \gamma\omega_2^2(a_2 + a_2^\dagger) \end{aligned} \quad (22)$$

$$H_{\text{PU}} = 2\gamma(\omega_1^2 - \omega_2^2)(\omega_1^2 a_1^\dagger a_1 - \omega_2^2 a_2^\dagger a_2) + (\omega_1 + \omega_2)/2 \quad (23)$$

$$[a_1, a_1^\dagger] = \frac{1}{2\gamma\omega_1(\omega_1^2 - \omega_2^2)}, \quad [a_2, a_2^\dagger] = -\frac{1}{2\gamma\omega_2(\omega_1^2 - \omega_2^2)} \quad (24)$$

$$\begin{aligned} a_1|\Omega\rangle = a_2|\Omega\rangle = 0, \quad H_{\text{PU}}|\Omega\rangle &= \frac{1}{2}(\omega_1 + \omega_2)|\Omega\rangle, \\ \langle\Omega|a_2a_2^\dagger|\Omega\rangle &< 0, \end{aligned} \quad (25)$$

**Negative norm state problem looks insurmountable, but.....**

**QUANTUM MECHANICS IS A GLOBAL THEORY. NEED TO SUPPLY GLOBAL INFORMATION. NEED TO LOOK AT WAVE FUNCTIONS. FIND THAT  $H_{\text{PU}}$ ,  $z$ ,  $p_z$  ARE NOT HERMITIAN**

$$[x, p_x] = i, \quad p_x = -i \frac{\partial}{\partial x}, \quad [z, p_z] = i, \quad p_z = -i \frac{\partial}{\partial z} \quad (26)$$

$$\psi_0(z, x) = \exp \left[ \frac{\gamma}{2} (\omega_1 + \omega_2) \omega_1 \omega_2 z^2 + i \gamma \omega_1 \omega_2 z x - \frac{\gamma}{2} (\omega_1 + \omega_2) x^2 \right] \quad (27)$$

**The states of negative norm are also states of INFINITE norm since  $\int dx dz \psi_0^*(z, x) \psi_0(z, x)$  is divergent, and when acting on such states, one CANNOT set  $p_z = -i \partial / \partial z$**

$$\left[ e^{i\theta} z, -\frac{i}{e^{i\theta}} \frac{\partial}{\partial z} \right] \psi(e^{i\theta} z) = i \psi(e^{i\theta} z), \quad z \rightarrow -iz, \quad p_z \rightarrow \frac{\partial}{\partial z} \quad (28)$$

**$p_z$  and  $z$  not Hermitian – they are anti-Hermitian.**

$$y = e^{\pi p_z z / 2} z e^{-\pi p_z z / 2} = -iz, \quad q = e^{\pi p_z z / 2} p_z e^{-\pi p_z z / 2} = ip_z \quad (29)$$

$$H = \frac{p^2}{2\gamma} - iqx + \frac{\gamma}{2} (\omega_1^2 + \omega_2^2) x^2 + \frac{\gamma}{2} \omega_1^2 \omega_2^2 y^2 \neq H^\dagger, \quad p = p_x \quad (30)$$

$$\text{Hermitian} \quad [x, p] = i, \quad \text{Hermitian} \quad [y, q] = i, \quad \text{non-Hermitian} \quad H \quad (31)$$

The Hamiltonian is not Hermitian – but it is PT symmetric, and thus still has real eigenvalues. Bender and collaborators showed that  $H = p^2 + ix^3$  has a completely real energy spectrum.

$$C^2 = 1, \quad [C, PT] = 0, \quad [C, H] = 0, \quad C = e^Q P \quad (32)$$

$$Q = \alpha[pq + \gamma^2 \omega_1^2 \omega_2^2 xy], \quad \alpha = \frac{1}{\gamma \omega_1 \omega_2} \log \left( \frac{\omega_1 + \omega_2}{\omega_1 - \omega_2} \right) \quad (33)$$

$$\tilde{H} = e^{-Q/2} H e^{Q/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma \omega_1^2} + \frac{\gamma}{2} \omega_1^2 x^2 + \frac{\gamma}{2} \omega_1^2 \omega_2^2 y^2 \quad (34)$$

Hamiltonian can be diagonalized by a similarity transformation which is non-unitary since  $Q$  is Hermitian rather than anti-Hermitian. Original Hamiltonian  $H$  is thus a Hermitian Hamiltonian as written in a skew basis. The eigenstates of  $H$  and  $\tilde{H}$  are not unitarily equivalent, and thus ....

## ....THE NORM IS NOT THE DIRAC NORM

$$\tilde{H} = e^{-Q/2} H e^{Q/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma}{2}\omega_1^2 x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2 y^2 \quad (35)$$

$$\tilde{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle, \quad H|n\rangle = E_n|n\rangle, \quad |n\rangle = e^{Q/2}|\tilde{n}\rangle \quad (36)$$

$$\langle\tilde{n}|\tilde{H} = E_n\langle\tilde{n}|, \quad \langle n| \equiv \langle\tilde{n}|e^{Q/2}, \quad \langle n|e^{-Q}H = \langle n|e^{-Q}E_n \quad (37)$$

The energy eigenbra  $\langle n|e^{-Q} = \langle n|PC$  is not the Dirac conjugate of the energy eigenket  $|n\rangle$ , since  $\langle n|H^\dagger = \langle n|E_n$  is not an eigenvalue equation for  $H$ .

$$\langle\tilde{n}|\tilde{m}\rangle = \delta_{m,n}, \quad \Sigma|\tilde{n}\rangle\langle\tilde{n}| = \mathbf{1}, \quad \tilde{H} = \Sigma|\tilde{n}\rangle E_n \langle\tilde{n}| \quad (38)$$

$$\langle n|e^{-Q}|m\rangle = \delta_{m,n}, \quad \Sigma|n\rangle\langle n|e^{-Q} = \mathbf{1}, \quad H = \Sigma|n\rangle E_n \langle n|e^{-Q} \quad (39)$$

The  $e^{-Q}$  norm is positive and so theory is unitary. Since  $C^2 = 1$ , its eigenvalues are  $\pm 1$ , with the relative plus and minus signs in the fourth-order propagator being due to the fact that the two poles have opposite signed eigenvalues of  $C$ .

# 1 NON-HERMITICITY AND UNITARITY

$$i\frac{d}{dt}|R_S(t)\rangle = H|R_S(t)\rangle, \quad -i\frac{d}{dt}\langle R_S(t)| = \langle R_S(t)|H^\dagger \quad (40)$$

$$|R_S(t)\rangle = e^{-iHt}|R_S(0)\rangle, \quad \langle R_S(t)| = \langle R_S(0)|e^{iH^\dagger t} \quad (41)$$

$$\langle R_S(t)|R_S(t)\rangle = \langle R_S(0)|e^{iH^\dagger t}e^{-iHt}|R_S(0)\rangle \neq \langle R_S(0)|R_S(0)\rangle \quad \text{NOT UNITARY} \quad (42)$$

$$\langle R_S(t)|A_S|R_S(t)\rangle = \langle R_S(0)|e^{iH^\dagger t}A_S e^{-iHt}|R_S(0)\rangle \quad (43)$$

$$A_H(t) = e^{iH^\dagger t}A_S e^{-iHt} \quad (44)$$

$$\text{OBTAIN} \quad i\frac{d}{dt}A_H(t) = A_H(t)H - H^\dagger A_H(t), \quad i\frac{d}{dt}A_H(t) = A_H(t)H - HA_H(t) \quad \text{NEEDED} \quad (45)$$

$$i\frac{d}{dt}|R_S(t)\rangle = H|R_S(t)\rangle, \quad -i\frac{d}{dt}\langle L_S(t)| = \langle L_S(t)|H \quad (46)$$

$$|R_S(t)\rangle = e^{-iHt}|R_S(0)\rangle, \quad \langle L_S(t)| = \langle L_S(0)|e^{iHt} \quad (47)$$

## LEFT-EIGENVECTOR IS NOT DIRAC CONJUGATE OF RIGHT EIGENVECTOR

$$\langle L_S(t)|R_S(t)\rangle = \langle L_S(0)|e^{iHt}e^{-iHt}|R_S(0)\rangle = \langle L_S(0)|R_S(0)\rangle \quad \text{UNITARY} \quad (48)$$

$$H^\dagger = e^{-Q}He^Q, \quad \langle R|H^\dagger = \langle R|E, \quad \langle R|e^{-Q}H = \langle R|e^{-Q}E, \quad \langle L| = \langle R|e^{-Q}, \quad \sum |R\rangle\langle L| = \sum |R\rangle\langle R|e^{-Q} = 1 \quad (49)$$

# NON-DIAGONALIZABILITY AND UNITARITY THE SINGULAR EQUAL-FREQUENCY LIMIT

In equal frequency limit the diagonalizing operator  $Q$  becomes singular and partial fraction decomposition of propagator becomes undefined.

$$\begin{aligned} \psi_0(x, y, t) &= \exp \left[ -\frac{\gamma}{2}(\omega_1 + \omega_2)(x^2 + \omega_1\omega_2 y^2) - \gamma\omega_1\omega_2 yx \right] \exp(-iE_0 t), \\ E_0 &= (\omega_1 + \omega_2)/2 \end{aligned} \quad (50)$$

$$\begin{aligned} \psi_1(x, y, t) &= (x + \omega_2 y)\psi_0(x, y, t)e^{-i\omega_1 t}, & E_1 &= E_0 + \omega_1 \\ \psi_2(x, y, t) &= (x + \omega_1 y)\psi_0(x, y, t)e^{-i\omega_2 t}, & E_2 &= E_0 + \omega_2 \end{aligned} \quad (51)$$

$$\hat{\psi}_0(x, y, t) = \exp \left[ -\gamma\omega^3 y^2 - \gamma\omega^2 yx - \gamma\omega x^2 - i\omega t \right], \quad \hat{E}_0 = \omega \quad (52)$$

$$\hat{\psi}_1(x, y, t) = (x + \omega y)\hat{\psi}_0(x, y, t)e^{-i\omega t}, \quad \hat{E}_1 = \hat{E}_0 + \omega \quad (53)$$

**TWO** one-particle states have collapsed into **ONE** state.

## THE MISSING ENERGY EIGENSTATES....

$$H_{1P}(\epsilon) = \frac{1}{2\omega} \begin{pmatrix} 4\omega^2 + \epsilon^2 & 4\omega^2 - \epsilon^2 \\ \epsilon^2 & 4\omega^2 - \epsilon^2 \end{pmatrix}, \quad \omega_1 \equiv \omega + \epsilon, \quad \omega_2 \equiv \omega - \epsilon, \quad (54)$$

$$|2\omega + \epsilon\rangle \equiv \begin{pmatrix} 2\omega + \epsilon \\ \epsilon \end{pmatrix}, \quad |2\omega - \epsilon\rangle \equiv \begin{pmatrix} 2\omega - \epsilon \\ -\epsilon \end{pmatrix} \quad (55)$$

$$S^{-1} \begin{pmatrix} 1 \\ 2\omega \end{pmatrix} \begin{pmatrix} 4\omega^2 + \epsilon^2 & 4\omega^2 - \epsilon^2 \\ \epsilon^2 & 4\omega^2 - \epsilon^2 \end{pmatrix} S = \begin{pmatrix} 2\omega + \epsilon & 0 \\ 0 & 2\omega - \epsilon \end{pmatrix} \quad (56)$$

$$S = \frac{1}{2\epsilon\omega^{1/2}(2\omega + \epsilon)^{1/2}} \begin{pmatrix} 2\omega + \epsilon & -(4\omega^2 - \epsilon^2)\epsilon \\ \epsilon & (2\omega + \epsilon)\epsilon^2 \end{pmatrix} \quad (57)$$

$$H_{1P}(\epsilon = 0) = 2\omega \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (58)$$

**Non-diagonalizable, Jordan-block matrix with TWO eigenvalues ( $\lambda_1 = 1$ ,  $\lambda_2 = 1$  since  $Tr = 1$ ,  $Det = 1$ ), but only ONE eigenvector.**

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c + d \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}, \quad d = 0 \quad (59)$$

## ... BECAME NONSTATIONARY

$$\begin{aligned}\hat{\psi}_{1a}(x, y, t) &= \lim_{\epsilon \rightarrow 0} \frac{\psi_2(x, y, t) - \psi_1(x, y, t)}{2\epsilon} \\ &= [(x + \omega y)it + y] \hat{\psi}_0(x, y, t) e^{-i\omega t}\end{aligned}\quad (60)$$

$$i \frac{\partial}{\partial t} \hat{\psi}(x, y, t) = \left( -\frac{1}{2\gamma} \frac{\partial^2}{\partial x^2} - x \frac{\partial}{\partial y} + \gamma \omega^2 x^2 + \frac{\gamma}{2} \omega^4 y^2 \right) \hat{\psi}(x, y, t) \quad (61)$$

Stationary plus non-stationary together are complete since just the right number of independent polynomial functions of  $x$  and  $y$ .

$$i \frac{\partial}{\partial t} \int dx dy \hat{\psi}_B^c(x, y, t) \hat{\psi}_A(x, y, t) = - \int dx dy x \frac{\partial}{\partial y} [\hat{\psi}_B^c(x, y, t) \hat{\psi}_A(x, y, t)] \quad (62)$$

$$i \frac{\partial}{\partial t} \int dx dy \hat{\psi}_B^c(x, y, t) \hat{\psi}_A(x, y, t) = 0 \quad (63)$$

Norm preserved in time so time evolution is unitary.

# Conformal Supergravity in Twistor-String Theory

N. Berkovits and E. Witten

June 2004 (arXiv:hep-th/040605). JHEP 0408 (2004) 009

“The net effect is that the translation generator  $D$  acts as

$$\begin{pmatrix} P & * \\ 0 & P \end{pmatrix}$$

where  $P$  would represent ordinary translations and the off-diagonal  $*$  arises from  $[D, \partial_I] \neq 0$ .

This matrix is not diagonalizable. This clashes with our usual experience. We are accustomed to the idea that the translation generators are Hermitian operators and so can be diagonalized. However, conformal supergravity is not a unitary theory, and one symptom of this is that the translation generators are undiagonalizable.”

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Bender and Mannheim: Not so fast.

## WHAT IS SO SPECIAL ABOUT FOURTH-ORDER THEORIES TO CAUSE ALL THIS

Consider the Lehmann Representation for a scalar field. Assume translation invariance and a bounded, real energy eigenspectrum with states of 4-momentum  $k_\mu^n$  with  $k_0^n > 0$ . We can set  $\phi(x) = e^{iP \cdot x} \phi(0) e^{-iP \cdot x}$ . Provisionally set  $\sum |n\rangle \langle n| = 1$ . Then can show

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \sum_n |\langle \Omega | \phi(0) | k_n^\mu \rangle|^2 e^{-ik^n \cdot (x-y)} \quad (64)$$

Now introduce the spectral function

$$\rho(q^2) = (2\pi)^3 \sum_n \delta^4(k_\mu^n - q_\mu) |\langle \Omega | \phi(0) | k_n^\mu \rangle|^2 \theta(q_0) \quad (65)$$

and set

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \int_0^\infty dm^2 \rho(m^2) \int \frac{d^4 q}{(2\pi)^3} \theta(q_0) \delta(q^2 - m^2) e^{-iq \cdot (x-y)} \quad (66)$$

Thus finally obtain the Lehmann representation

$$\langle \Omega | T[\phi(x) \phi(y)] | \Omega \rangle = \int_0^\infty dm^2 \rho(m^2) \Delta_2^{\text{free}}(x - y; m^2) \quad (67)$$

This relation holds for any interacting two-point function no matter what its equation of motion, with it always being the **FREE SECOND-ORDER** Feynman propagator which appears in the integral because the mass shell condition  $p^2 = m^2$  is always second-order. The Lehmann representation thus holds in fourth-order theories also. However, for large  $k^2$  the second-order Feynman propagator behaves as  $1/k^2$ , whereas for fourth-order theories the propagator behaves as  $1/k^4$ . Hence we have a contradiction if  $\rho(m^2)$  is positive definite.

Solution is that spectral function cannot be positive definite, and we cannot set  $\Sigma |n\rangle\langle n| = 1$ . Rather we must set  $\Sigma |n\rangle\langle n|e^{-Q} = 1$  (and not  $\Sigma |n\rangle\langle n| - \Sigma |m\rangle\langle m| = 1$ ), and distinguish between left and right momentum eigenvectors. Thus must use  $|R\rangle$  and  $\langle L| = \langle R|e^{-Q}$ . With this choice, the spectral function is replaced by

$$\rho(q^2) = (2\pi)^3 \sum_n \delta^4(k_\mu^n - q_\mu) \langle \Omega | e^{-Q} \phi(0) | k_n^\mu \rangle \langle k_n^\mu | e^{-Q} \phi(0) | \Omega \rangle \theta(q_0) \quad (68)$$

Now there is no positivity requirement and we can cancel the  $1/k^2$  behavior without having to give up unitarity. Since we still impose the reality of the momentum eigenvalues, the Hamiltonian of the theory must be  $PT$  invariant rather than Hermitian. From the Lehmann representation we thus conclude that the Hamiltonian of theories such as the fourth-order Pais-Uhlenbeck oscillator cannot be Hermitian and must instead be  $PT$  invariant, just as we had found directly.

## THE REMARKABLE MORAL OF THE STORY

Consider **ANY** higher derivative theory in which the equation of motion is of the form  $f(D)\phi = 0$  where  $D = \partial_\mu \partial^\mu$  and  $f(D) = \Sigma a_n D^n$ . Also require that all momentum eigenvalues be real. In such a theory, at large  $k^2$  the propagator will behave as  $1/k^{2n}$ . Hence there will be a contradiction with the Lehmann representation if we require the standard  $\Sigma |n\rangle\langle n| = 1$ . Rather, such theories must be  $PT$  theories rather than standard Hermitian ones. Hence once we depart from second-order equations we are forced to  $PT$ -invariant, non-Hermitian Hamiltonians.  $PT$ -invariance is thus the general rule, and it is only a historical accident (second-order theories were encountered first) that it was not discovered earlier.

# DYNAMICAL SYMMETRY BREAKING AND THE COSMOLOGICAL CONSTANT PROBLEM

**The zero-point energy and cosmological constant problems are two separate problems. With dynamical symmetry breaking they solve each other.**

The zero-point energy problem already exists in a free field theory, and as such is separate from any cosmological constant term that might be induced by spontaneous symmetry breaking. The cosmological constant is associated with the minimum of the symmetry breaking potential while the zero-point energy is associated with the fluctuations about it. Moreover, the zero-point term and the cosmological constant term even transform differently under a general coordinate transformation, the former possessing a fluid velocity and being maximally 3-symmetric, with the latter possessing no fluid velocity and being maximally 4-symmetric. The zero-point fluctuation term is associated with a perfect matter fluid in which both  $\rho_m$  and  $p_m$  are positive, – so that  $T_{\mu\nu} = (\rho_m + p_m)U_\mu U_\nu + p_m g_{\mu\nu}$ . While the cosmological constant term is associated with a perfect fluid in which  $p = -\rho$ , – so that  $T_{\mu\nu} = -\Lambda g_{\mu\nu}$ .

**When the symmetry is broken dynamically by fermion condensates, it is the fermionic zero-point fluctuations which cause the change in the vacuum in the first place, to thus actually produce the cosmological constant. In this case the zero-point and cosmological constant terms are not independent, and are related in a way which allows each one to cancel the other, so that both the zero-point and cosmological constant problems solve each other.**

For a free quantum fermion field, in a mode of the form

$$\psi(x) = \sum_{\pm s} \int \frac{d^4 p}{(2\pi)^{3/2}} \left( \frac{m}{E_p} \right)^{1/2} [b(p, s)u(p, s)e^{-ip \cdot x} + d^\dagger(p, s)v(p, s)e^{ip \cdot x}] \quad (69)$$

the vacuum expectation value of  $T_{\mu\nu} \sim \bar{\psi}\gamma_\mu\partial_\nu\psi$  is due to the non-vanishing of

$$\langle \Omega | d(p, s) d^\dagger(p, s) | \Omega \rangle = \langle \Omega | (d(p, s) d^\dagger(p, s) + d^\dagger(p, s) d(p, s)) | \Omega \rangle, \quad (70)$$

to yield the zero-point fluctuation contribution

$$\begin{aligned} \langle \Omega | T_{\mu\nu} | \Omega \rangle &\sim -2 \int \frac{d^3 p}{(2\pi)^3} \frac{p_\mu p_\nu}{E_p}, & \rho_m = \langle \Omega | T_{00} | \Omega \rangle &\sim -2 \int \frac{d^3 p}{(2\pi)^3} E_p, \\ p_m = \langle \Omega | T_{xx} | \Omega \rangle &= \langle \Omega | T_{yy} | \Omega \rangle = \langle \Omega | T_{zz} | \Omega \rangle &\sim -2 \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{3E_p}. \end{aligned} \quad (71)$$

and a perfect fluid form  $T_{\mu\nu} = (\rho_m + p_m)U_\mu U_\nu + p_m g_{\mu\nu}$  where  $\rho_m$  and  $p_m$  have the **SAME** sign. That this is **NOT** a perfect fluid with  $p_m = -\rho_m$  is because the averaging is only 3-dimensional, i.e. for modes which obey the wave equation only the 3-momentum is averaged over. Moreover, the velocity of the fluid  $U^\mu$  is a timelike vector, and is allowed simply because a Minkowski metric distinguishes between timelike and spacelike.

In flat space, for  $T_{\mu\nu} \sim k_\mu k_\nu / E_k$  one can also construct the total energy and momentum as  $P_\mu = \int d^3 x T_{0\mu}$ , to find that

$$P_0 \sim E_k, \quad P_i = 0, \quad (72)$$

with there being no zero-point total momentum. However one can also evaluate

$$\int d^3x T_{ij} \sim \int d^3x \frac{k_i k_j}{E_k} \sim \delta_{ij} \quad (73)$$

to thus yield a zero-point quadrupole pressure tensor. Thus the absence of a zero-point total momentum does not mean the absence of a zero-point pressure; and while such a term plays no role in flat space it still couples to gravity in curved space.

To underscore the fact that the zero-point term is not the same as a cosmological constant term, we note that for a free massless field where  $p^\mu p_\mu = 0$ , the energy-momentum tensor is traceless and  $\rho_m = 3p_m$ , something not possible for a cosmological constant term, since a traceless  $\Lambda g_{\mu\nu}$  would require  $\Lambda = 0$ .

A cosmological constant term is induced when the symmetry is broken and adds a term of the form of  $T_{\mu\nu} = -\Lambda g_{\mu\nu}$ . It yields a contribution to the trace of the form  $T^\mu{}_\mu = -4\Lambda$ , with the zero-point energy and the cosmological constant terms thus being different. In the presence of both the energy density is given by  $T_{00} = \rho_m + \Lambda$ .

When mass is generated dynamically in a four-Fermi theory, the mean field is produced by zero-point fluctuations. The cosmological constant is thus induced by zero-point fluctuations also and can thus be related to the zero-point energy. Since the trace of  $T_{\mu\nu}$  is zero in a conformal invariant theory,  $3p_m - \rho_m - 4\Lambda = 0$ , and the zero-point and cosmological constant terms cancel each other identically. Neither can be bigger or smaller than the other.

## FOUR-FERMI MEAN-FIELD THEORY IN TWO DIMENSIONS

In  $D = 2$  the four-Fermi theory is defined via

$$L = i\bar{\psi}\gamma^\mu\partial_\mu\psi - (g/2)[\bar{\psi}\psi]^2, \quad i\gamma^\mu\partial_\mu\psi - g[\bar{\psi}\psi]\psi = 0, \quad (74)$$

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\frac{g}{2}[\bar{\psi}\psi]^2, \quad T^\mu{}_\mu = i\bar{\psi}\gamma^\mu\partial_\mu\psi - g[\bar{\psi}\psi]^2 = 0, \quad (75)$$

with the energy-momentum tensor being traceless. In the mean-field approximation one looks for self-consistent, translation invariant, states  $|S\rangle$  in which

$$\langle S|\bar{\psi}\psi|S\rangle = m/g, \quad \langle S|[\bar{\psi}\psi - m/g]^2|S\rangle = 0, \quad i\gamma^\mu\partial_\mu\psi - m\psi = 0, \quad (76)$$

$$\langle S|T^{\mu\nu}|S\rangle = \langle S|i\bar{\psi}\gamma^\mu\partial^\nu\psi|S\rangle - g^{\mu\nu}\frac{m^2}{2g}, \quad \langle S|T^\mu{}_\mu|S\rangle = m\langle S|\bar{\psi}\psi|S\rangle - \frac{m^2}{g} = 0. \quad (77)$$

with the mean-field approximation preserving tracelessness.

In a plane wave solution

$$\psi(x) = \sum_{\pm s} \int d^2p/(2\pi)^{1/2}(m/E_p)^{1/2}[b(p, s)u(p, s)e^{-ip\cdot x} + d^\dagger(p, s)v(p, s)e^{ip\cdot x}] \quad (78)$$

we find that

$$\langle \Omega|i\bar{\psi}\gamma^\mu\partial^\nu\psi|\Omega\rangle = -2 \int d^2pp^\mu p^\nu / (2\pi E_p) \quad (79)$$

due to zero-point fluctuations since

$$\langle \Omega|bb^\dagger|\Omega\rangle = \langle \Omega|(bb^\dagger + b^\dagger b)|\Omega\rangle. \quad (80)$$

## CANCELLATION OF ZERO-POINT AND $\Lambda$ IN TRACE OF $T_{\mu\nu}$

Incoherently adding together modes with  $p^\mu = (E_p, p)$  and  $(E_p, -p)$  gives

$$\Sigma \frac{p^\mu p^\nu}{E_p} = \begin{pmatrix} E_p & p \\ p & p^2/E_p \end{pmatrix} + \begin{pmatrix} E_p & -p \\ -p & p^2/E_p \end{pmatrix} = \begin{pmatrix} 2E_p & 0 \\ 0 & 2p^2/E_p \end{pmatrix} = (\rho_m + p_m)U^\mu U^\nu + p_m g^{\mu\nu} \quad (81)$$

On setting  $m^2/g = g[\langle S|\bar{\psi}\psi|S\rangle]^2 = \Lambda$ , we can thus set

$$\langle S|T^{\mu\nu}|S\rangle = (\rho_m + p_m)U^\mu U^\nu + p_m g^{\mu\nu} - g^{\mu\nu} \Lambda, \quad \langle S|T^\mu{}_\mu|S\rangle = p_m - \rho_m - 2\Lambda = 0. \quad (82)$$

Thus  $2\Lambda = p_m - \rho_m$ , neither bigger nor smaller. The reason for this is that all of  $\Lambda$ ,  $\rho_m$  and  $p_m$  are determined in one the same state  $|S\rangle$ . For the case of a fundamental scalar field  $\phi(x)$ ,  $\Lambda = \lambda\phi^4$  is determined by the vacuum (location of the minimum of the Higgs double-well potential), while  $\rho_m$  and  $p_m$  are determined by whichever matter field frequency modes are occupied. Hence one cannot relate  $\Lambda$  to  $\rho_m$  and  $p_m$  in standard cosmology with a fundamental Higgs field, to thus give rise to the cosmological constant problem. If however scalar field is a c-number Ginzburg-Landau condensate  $\langle S|\bar{\psi}\psi|S\rangle$  then can relate  $\Lambda$ ,  $\rho_m$  and  $p_m$ , and even have them cancel each other.

**Comparison test:** If Higgs is a c-number condensate then will **NOT** be produced in an accelerator such as the LHC, while if Higgs is fundamental then will be produced in an accelerator.

# CANCELLATION OF ZERO-POINT AND $\Lambda$ IN $T_{\mu\nu}$ ITSELF

In four spacetime dimensions invariance under

$$g_{\mu\nu}(x) \rightarrow e^{2\alpha(x)} g_{\mu\nu}(x), \quad (83)$$

leads to a unique gravitational action

$$I_W = -\alpha_g \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha_g \int d^4x (-g)^{1/2} \left[ R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} (R^\alpha{}_\alpha)^2 \right] \quad (84)$$

where  $\alpha_g$  is dimensionless and  $C_{\lambda\mu\nu\kappa}$  is the conformal Weyl tensor. The associated gravitational equations of motion are the **fourth-order derivative**:

$$4\alpha_g [2C^{\mu\lambda\nu\kappa}{}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}] = T^{\mu\nu} \quad (85)$$

and thus in cosmology where  $C_{\lambda\mu\nu\kappa} = 0$ , we obtain

$$T^{\mu\nu} = 0 \quad (86)$$

Thus again get cancellation of zero-point and  $\Lambda$  terms in  $T_{\mu\nu}$  itself, but now need to include zero-point fluctuations in gravitational field as well. While  $\langle S|[2C^{\mu\lambda\nu\kappa}{}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}]|S \rangle$  vanishes in a classical cosmological background, quantum-mechanically there is a zero-point fluctuation contribution. Using the gravitational zero-point fluctuations to cancel the matter field zero-point fluctuations and  $\Lambda$  is only achievable in a renormalizable theory of gravity. Hence works in conformal gravity but not in standard gravity.

## 2 FOURTH ORDER UNITARITY PROBLEM

$$I = \frac{1}{2} \int d^4x \left[ \partial_\mu \partial_\nu \phi \partial^\mu \partial^\nu \phi - M^2 \partial_\mu \phi \partial^\mu \phi \right] \quad (87)$$

$$(\partial_t^2 - \nabla^2)(\partial_t^2 - \nabla^2 + M^2)\phi(\bar{x}, t) = 0 \quad (88)$$

$$D^{(4)}(k^2) = \frac{1}{k^2(k^2 - M^2)} = \frac{1}{M^2} \left( \frac{1}{k^2 - M^2} - \frac{1}{k^2} \right) \quad (89)$$

$$\Delta_{\text{F}}^{\text{int}}(x - y) = i \langle \Omega | T[\phi(x)\phi(y)] | \Omega \rangle \quad (90)$$

$$\sum_n |n\rangle \langle n| = \mathbf{1} \quad (91)$$

$$\rho(q^2) = (2\pi)^3 \sum_n \delta^4(k_\mu^n - q_\mu) |\langle \Omega | \phi(0) | k_\mu^n \rangle|^2 \theta(q_0) \quad (92)$$

$$\Delta_{\text{F}}^{\text{int}}(x - y) = \int_0^\infty dm^2 \rho(m^2) \Delta_{(\text{F},2)}^{\text{free}}(x - y; m^2) \quad (93)$$

$$D(\bar{x}, \bar{x}', E) = \sum_n \frac{\psi_n(\bar{x}) \psi_n^*(\bar{x}')}{E - E_n} \quad (94)$$

$$\sum_n |n\rangle \langle n| - \sum_m |m\rangle \langle m| = \mathbf{1} \quad (95)$$

### 3 THE PAIS-UHLENBECK OSCILLATOR

$$\phi(\bar{x}, t) \sim z(t)e^{i\bar{k}\cdot\bar{x}}, \quad \omega_1 = (\bar{k}^2 + M^2)^{1/2}, \quad \omega_2 = |\bar{k}| \quad (96)$$

$$\frac{d^4 z}{dt^4} + (\omega_1^2 + \omega_2^2) \frac{d^2 z}{dt^2} + \omega_1^2 \omega_2^2 z = 0 \quad (97)$$

$$I_{\text{PU}} = \frac{\gamma}{2} \int dt [\ddot{z}^2 - (\omega_1^2 + \omega_2^2) \dot{z}^2 + \omega_1^2 \omega_2^2 z^2] \quad (98)$$

$$H_{\text{PU}} = \frac{p_x^2}{2\gamma} + p_z x + \frac{\gamma}{2} (\omega_1^2 + \omega_2^2) x^2 - \frac{\gamma}{2} \omega_1^2 \omega_2^2 z^2, \quad x = \dot{z} \quad (99)$$

$$[x, p_x] = i, \quad [z, p_z] = i \quad (100)$$

$$\begin{aligned} z &= a_1 + a_1^\dagger + a_2 + a_2^\dagger, & p_z &= i\gamma\omega_1\omega_2^2(a_1 - a_1^\dagger) + i\gamma\omega_1^2\omega_2(a_2 - a_2^\dagger), \\ x &= -i\omega_1(a_1 - a_1^\dagger) - i\omega_2(a_2 - a_2^\dagger), & p_x &= -\gamma\omega_1^2(a_1 + a_1^\dagger) - \gamma\omega_2^2(a_2 + a_2^\dagger) \end{aligned} \quad (101)$$

$$H_{\text{PU}} = 2\gamma(\omega_1^2 - \omega_2^2)(\omega_1^2 a_1^\dagger a_1 - \omega_2^2 a_2^\dagger a_2) + (\omega_1 + \omega_2)/2 \quad (102)$$

$$[a_1, a_1^\dagger] = \frac{1}{2\gamma\omega_1(\omega_1^2 - \omega_2^2)}, \quad [a_2, a_2^\dagger] = -\frac{1}{2\gamma\omega_2(\omega_1^2 - \omega_2^2)} \quad (103)$$

$$\begin{aligned} a_1|\Omega\rangle = a_2|\Omega\rangle = 0, \quad H_{\text{PU}}|\Omega\rangle &= \frac{1}{2}(\omega_1 + \omega_2)|\Omega\rangle, \quad \langle\Omega|a_2 a_2^\dagger|\Omega\rangle < 0, \\ a_1|\hat{\Omega}\rangle = a_2^\dagger|\hat{\Omega}\rangle = 0, \quad H_{\text{PU}}|\hat{\Omega}\rangle &= \frac{1}{2}(\omega_1 - \omega_2)|\hat{\Omega}\rangle, \quad \langle\hat{\Omega}|a_2^\dagger a_2|\hat{\Omega}\rangle > 0 \end{aligned} \quad (104)$$

#### 4 SOLUTION: HAMILTONIAN NOT HERMITIAN – BUT IT IS PT SYMMETRIC

$$p_x = -i\frac{\partial}{\partial x}, \quad p_z = -i\frac{\partial}{\partial z} \quad (105)$$

$$\psi_0(z, x) = \exp \left[ \frac{\gamma}{2}(\omega_1 + \omega_2)\omega_1\omega_2 z^2 + i\gamma\omega_1\omega_2 z x - \frac{\gamma}{2}(\omega_1 + \omega_2)x^2 \right] \quad (106)$$

$$[z, p_z] = i, \quad \left[ e^{i\theta} z, -\frac{i}{e^{i\theta}} \frac{\partial}{\partial z} \right] \psi(e^{i\theta} z) = i\psi(e^{i\theta} z) \quad (107)$$

$$y = e^{\pi p_z z/2} z e^{-\pi p_z z/2} = -iz, \quad q = e^{\pi p_z z/2} p_z e^{-\pi p_z z/2} = ip_z \quad (108)$$

$$H = \frac{p^2}{2\gamma} - iqx + \frac{\gamma}{2}(\omega_1^2 + \omega_2^2)x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2 y^2, \quad p = p_x \quad (109)$$

$$[p, x] = i, \quad [q, y] = i \quad (110)$$

$$C^2 = 1, \quad [C, PT] = 0, \quad [C, H] = 0, \quad C = e^Q P \quad (111)$$

$$Q = \alpha[pq + \gamma^2\omega_1^2\omega_2^2 xy], \quad \alpha = \frac{1}{\gamma\omega_1\omega_2} \log \left( \frac{\omega_1 + \omega_2}{\omega_1 - \omega_2} \right) \quad (112)$$

$$\tilde{H} = e^{-Q/2} H e^{Q/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma}{2}\omega_1^2 x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2 y^2 \quad (113)$$

## 5 THE NORM IS NOT THE DIRAC NORM – IT IS THE PT NORM

$$\tilde{H} = e^{-Q/2} H e^{Q/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma}{2}\omega_1^2 x^2 + \frac{\gamma}{2}\omega_1^2 \omega_2^2 y^2 \quad (114)$$

$$\tilde{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle, \quad H|n\rangle = E_n|n\rangle, \quad |n\rangle = e^{Q/2}|\tilde{n}\rangle \quad (115)$$

$$\langle\tilde{n}|\tilde{H} = E_n\langle\tilde{n}|, \quad \langle n| \equiv \langle\tilde{n}|e^{Q/2}, \quad \langle n|e^{-Q}H = \langle n|e^{-Q}E_n \quad (116)$$

$$\langle\tilde{n}|\tilde{m}\rangle = \delta_{m,n}, \quad \sum_n |\tilde{n}\rangle\langle\tilde{n}| = \mathbf{1}, \quad \tilde{H} = \sum_n |\tilde{n}\rangle E_n \langle\tilde{n}| \quad (117)$$

$$\langle n|e^{-Q}|m\rangle = \delta_{m,n}, \quad \sum_n |n\rangle\langle n|e^{-Q} = \mathbf{1}, \quad H = \sum_n |n\rangle E_n \langle n|e^{-Q} \quad (118)$$

$$\sum_n |n\rangle\langle n|PC = \mathbf{1}, \quad H = \sum_n |n\rangle E_n \langle n|PC, \quad C_n = \pm 1 \quad (119)$$

$$\langle x, y|e^{-iHt}|x', y'\rangle = \sum_n \psi_n(x, y) C_n e^{-iE_n t} \psi_n^c(x', y') \quad (120)$$

## 6 NON-HERMITICITY AND UNITARITY

$$i\frac{d}{dt}|\alpha_S(t)\rangle = H|\alpha_S(t)\rangle, \quad -i\frac{d}{dt}\langle\alpha_S(t)| = \langle\alpha_S(t)|H^\dagger \quad (121)$$

$$|\alpha_S(t)\rangle = e^{-iHt}|\alpha_S(0)\rangle, \quad \langle\alpha_S(t)| = \langle\alpha_S(0)|e^{iH^\dagger t} \quad (122)$$

$$\langle\alpha_S(t)|\alpha_S(t)\rangle = \langle\alpha_S(0)|e^{iH^\dagger t}e^{-iHt}|\alpha_S(0)\rangle \neq \langle\alpha_S(0)|\alpha_S(0)\rangle \quad (123)$$

$$\langle\alpha_S(t)|A_S|\alpha_S(t)\rangle = \langle\alpha_S(0)|e^{iH^\dagger t}A_S e^{-iHt}|\alpha_S(0)\rangle \quad (124)$$

$$A_H(t) = e^{iH^\dagger t}A_S e^{-iHt} \quad (125)$$

$$i\frac{d}{dt}A_H(t) = A_H(t)H - H^\dagger A_H(t), \quad i\frac{d}{dt}A_H(t) = A_H(t)H - HA_H(t) \quad (126)$$

$$i\frac{d}{dt}|\alpha_S(t)\rangle = H|\alpha_S(t)\rangle, \quad -i\frac{d}{dt}\langle\hat{\alpha}_S(t)| = \langle\hat{\alpha}_S(t)|H \quad (127)$$

$$|\alpha_S(t)\rangle = e^{-iHt}|\alpha_S(0)\rangle, \quad \langle\hat{\alpha}_S(t)| = \langle\hat{\alpha}_S(0)|e^{iHt} \quad (128)$$

$$\langle\hat{\alpha}_S(t)|\alpha_S(t)\rangle = \langle\hat{\alpha}_S(0)|e^{iHt}e^{-iHt}|\alpha_S(0)\rangle = \langle\hat{\alpha}_S(0)|\alpha_S(0)\rangle \quad (129)$$

$$\langle\hat{\alpha}_S(t)| = \langle\alpha_S(t)|e^{-Q}, \quad H^\dagger = e^{-Q}He^Q \quad (130)$$

## 7 NON-DIAGONALIZABILITY AND UNITARITY– EQUAL-FREQUENCY LIMIT IS SINGULAR

$$\psi_0(x, y, t) = \exp \left[ -\frac{\gamma}{2}(\omega_1 + \omega_2)(x^2 + \omega_1\omega_2y^2) - \gamma\omega_1\omega_2yx \right] \exp(-iE_0t), \quad E_0 = (\omega_1 + \omega_2)/2 \quad (131)$$

$$\begin{aligned} \psi_1(x, y, t) &= (x + \omega_2y)\psi_0(x, y, t)e^{-i\omega_1t}, & E_1 &= E_0 + \omega_1 \\ \psi_2(x, y, t) &= (x + \omega_1y)\psi_0(x, y, t)e^{-i\omega_2t}, & E_2 &= E_0 + \omega_2 \end{aligned} \quad (132)$$

$$\begin{aligned} \psi_3(x, y, t) &= [(x + \omega_2y)^2 - 1/2\gamma\omega_1] \psi_0(x, y, t)e^{-2i\omega_1t}, & E_3 &= E_0 + 2\omega_1 \\ \psi_4(x, y, t) &= [(x + \omega_1y)(x + \omega_2y) - 1/\gamma(\omega_1 + \omega_2)] \psi_0(x, y, t)e^{-i(\omega_1+\omega_2)t}, & E_4 &= E_0 + \omega_1 + \omega_2 \\ \psi_5(x, y, t) &= [(x + \omega_1y)^2 - 1/2\gamma\omega_2] \psi_0(x, y, t)e^{-2i\omega_2t}, & E_5 &= E_0 + 2\omega_2 \end{aligned} \quad (133)$$

$$\hat{\psi}_0(x, y, t) = \exp \left[ -\gamma\omega^3y^2 - \gamma\omega^2yx - \gamma\omega x^2 - i\omega t \right], \quad \hat{E}_0 = \omega \quad (134)$$

$$\hat{\psi}_1(x, y, t) = (x + \omega y)\hat{\psi}_0(x, y, t)e^{-i\omega t}, \quad \hat{E}_1 = \hat{E}_0 + \omega \quad (135)$$

$$\hat{\psi}_2(x, y, t) = [(x + \omega y)^2 - 1/2\gamma\omega] \hat{\psi}_0(x, y, t)e^{-2i\omega t}, \quad \hat{E}_2 = \hat{E}_0 + 2\omega \quad (136)$$

## 8 THE MISSING ENERGY EIGENSTATES....

$$H_{1P}(\epsilon) = \frac{1}{2\omega} \begin{pmatrix} 4\omega^2 + \epsilon^2 & 4\omega^2 - \epsilon^2 \\ \epsilon^2 & 4\omega^2 - \epsilon^2 \end{pmatrix} \quad (137)$$

$$|2\omega + \epsilon\rangle \equiv \begin{pmatrix} 2\omega + \epsilon \\ \epsilon \end{pmatrix}, \quad |2\omega - \epsilon\rangle \equiv \begin{pmatrix} 2\omega - \epsilon \\ -\epsilon \end{pmatrix} \quad (138)$$

$$S^{-1} \begin{pmatrix} 1 \\ \frac{1}{2\omega} \end{pmatrix} \begin{pmatrix} 4\omega^2 + \epsilon^2 & 4\omega^2 - \epsilon^2 \\ \epsilon^2 & 4\omega^2 - \epsilon^2 \end{pmatrix} S = \begin{pmatrix} 2\omega + \epsilon & 0 \\ 0 & 2\omega - \epsilon \end{pmatrix} \quad (139)$$

$$S = \frac{1}{2\epsilon\omega^{1/2}(2\omega + \epsilon)^{1/2}} \begin{pmatrix} 2\omega + \epsilon & -(4\omega^2 - \epsilon^2)\epsilon \\ \epsilon & (2\omega + \epsilon)\epsilon^2 \end{pmatrix},$$

$$S^{-1} = \frac{1}{2\epsilon\omega^{1/2}(2\omega + \epsilon)^{1/2}} \begin{pmatrix} (2\omega + \epsilon)\epsilon^2 & (4\omega^2 - \epsilon^2)\epsilon \\ -\epsilon & 2\omega + \epsilon \end{pmatrix} \quad (140)$$

$$H_{1P}(\epsilon = 0) = 2\omega \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (141)$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c + d \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \quad (142)$$

## 9 ... BECAME NONSTATIONARY

$$\omega_1 \equiv \omega + \epsilon, \quad \omega_2 \equiv \omega - \epsilon, \quad \epsilon \rightarrow 0 \quad (143)$$

$$\begin{aligned} \hat{\psi}_{1a}(x, y, t) &= \lim_{\epsilon \rightarrow 0} \frac{\psi_2(x, y, t) - \psi_1(x, y, t)}{2\epsilon} \\ &= [(x + \omega y)it + y] \hat{\psi}_0(x, y, t) e^{-i\omega t} \end{aligned} \quad (144)$$

$$\begin{aligned} \hat{\psi}_{2a}(x, y, t) &= \lim_{\epsilon \rightarrow 0} \frac{\psi_5(x, y, t) - \psi_3(x, y, t)}{2\epsilon} \\ &= \left[ \left( (x + \omega y)^2 - \frac{1}{2\gamma\omega} \right) 2it + 2xy + 2\omega y^2 - \frac{1}{2\gamma\omega^2} \right] \hat{\psi}_0(x, y, t) e^{-2i\omega t} \end{aligned} \quad (145)$$

$$\begin{aligned} \hat{\psi}_{2b}(x, y, t) &= \lim_{\epsilon \rightarrow 0} \frac{2\psi_4(x, y, t) - \psi_3(x, y, t) - \psi_5(x, y, t)}{2\epsilon^2} \\ &= \left[ \left( (x + \omega y)^2 - \frac{1}{2\gamma\omega} \right) 2t^2 - \left( 2xy + 2\omega y^2 - \frac{1}{2\gamma\omega^2} \right) 2it \right. \\ &\quad \left. - 2y^2 + \frac{1}{2\gamma\omega^3} \right] \hat{\psi}_0(x, y, t) e^{-2i\omega t} \end{aligned} \quad (146)$$

$$i \frac{\partial}{\partial t} \hat{\psi}(x, y, t) = \left( -\frac{1}{2\gamma} \frac{\partial^2}{\partial x^2} - x \frac{\partial}{\partial y} + \gamma\omega^2 x^2 + \frac{\gamma}{2} \omega^4 y^2 \right) \hat{\psi}(x, y, t) \quad (147)$$

$$i \frac{\partial}{\partial t} \int dx dy \hat{\psi}_B^c(x, y, t) \hat{\psi}_A(x, y, t) = - \int dx dy x \frac{\partial}{\partial y} \left[ \hat{\psi}_B^c(x, y, t) \hat{\psi}_A(x, y, t) \right] \quad (148)$$

$$i \frac{\partial}{\partial t} \int dx dy \hat{\psi}_B^c(x, y, t) \hat{\psi}_A(x, y, t) = 0 \quad (149)$$

# Conformal Supergravity in Twistor-String Theory

N. Berkovits and E. Witten

June 2004 (arXiv:hep-th/040605). JHEP 0408 (2004) 009

“The net effect is that the translation generator  $D$  acts as

$$\begin{pmatrix} P & * \\ 0 & P \end{pmatrix}$$

where  $P$  would represent ordinary translations and the off-diagonal  $*$  arises from  $[D, \partial_I] \neq 0$ .

This matrix is not diagonalizable. This clashes with our usual experience. We are accustomed to the idea that the translation generators are Hermitian operators and so can be diagonalized. However, conformal supergravity is not a unitary theory, and one symptom of this is that the translation generators are undiagonalizable.”

## 10 SHORTCOMINGS OF EINSTEIN GRAVITY

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad g^{\mu\lambda} g_{\mu\nu} = g^\lambda{}_\nu = \delta_\nu^\lambda \quad (150)$$

$$\Gamma_{\nu\sigma}^\mu = \frac{1}{2} g^{\mu\lambda} [\partial_\nu g_{\lambda\sigma} + \partial_\sigma g_{\lambda\nu} - \partial_\lambda g_{\nu\sigma}] \quad (151)$$

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\nu\sigma}^\mu \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} = 0 \quad (152)$$

$$R^\lambda{}_{\mu\nu\kappa} = \frac{\partial \Gamma_{\mu\nu}^\lambda}{\partial x^\kappa} + \Gamma_{\kappa\eta}^\lambda \Gamma_{\mu\nu}^\eta - \frac{\partial \Gamma_{\mu\kappa}^\lambda}{\partial x^\nu} - \Gamma_{\nu\eta}^\lambda \Gamma_{\mu\kappa}^\eta \quad (153)$$

$$R_{\mu\kappa} = g^\nu{}_\lambda R^\lambda{}_{\mu\nu\kappa}, \quad R^\mu{}_\mu = g^{\mu\kappa} R_{\mu\kappa} \quad (154)$$

$$I_{EH} = -\frac{1}{16\pi G} \int d^4x (-g)^{1/2} R^\alpha{}_\alpha \quad (155)$$

$$-\frac{1}{8\pi G} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha{}_\alpha \right) = T_{\mu\nu} \quad (156)$$

$$\nabla^2 \phi = \rho, \quad \phi = -\frac{\beta}{r} \quad (157)$$

$$\nabla^4 \phi = \rho, \quad \phi = -\frac{\beta}{r} + \frac{1}{2} \gamma r \quad (158)$$

## 11 CONFORMAL GRAVITY

$$g_{\mu\nu} \rightarrow e^{2\alpha(x)} g_{\mu\nu}, \quad C_{\lambda\mu\nu\kappa} \rightarrow e^{2\alpha(x)} C_{\lambda\mu\nu\kappa} \quad (159)$$

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} + \frac{1}{6} R^\alpha{}_\alpha [g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}] - \frac{1}{2} [g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu}] \quad (160)$$

$$I_W = -\alpha_g \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha_g \int d^4x (-g)^{1/2} \left[ R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} (R^\alpha{}_\alpha)^2 \right] \quad (161)$$

$$4\alpha_g W^{\mu\nu} = T^{\mu\nu} \quad (162)$$

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{2} g^{\mu\nu} (R^\alpha{}_\alpha)^{;\beta}{}_{;\beta} + R^{\mu\nu; \beta}{}_{;\beta} - R^{\mu\beta; \nu}{}_{;\beta} - R^{\nu\beta; \mu}{}_{;\beta} - 2R^{\mu\beta} R^\nu{}_\beta + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - \frac{2}{3} g^{\mu\nu} (R^\alpha{}_\alpha)^{;\beta}{}_{;\beta} \\ &+ \frac{2}{3} (R^\alpha{}_\alpha)^{;\mu; \nu} + \frac{2}{3} R^\alpha{}_\alpha R^{\mu\nu} - \frac{1}{6} g^{\mu\nu} (R^\alpha{}_\alpha)^2 = 2C^{\mu\lambda\nu\kappa}{}_{;\lambda; \kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa} \end{aligned} \quad (163)$$

$$-g_{00} = \frac{1}{g_{rr}} = 1 - \frac{2\beta}{r} + \gamma r \quad (164)$$

$$\Sigma(R) = \Sigma_0 e^{-R/R_0}, \quad N^* = 2\pi \Sigma_0 R_0^2 \quad (165)$$

$$\begin{aligned} \frac{v_{\text{tot}}^2}{R} &= \frac{N^* \beta^* c^2 R}{2R_0^3} \left[ I_0 \left( \frac{R}{2R_0} \right) K_0 \left( \frac{R}{2R_0} \right) - I_1 \left( \frac{R}{2R_0} \right) K_1 \left( \frac{R}{2R_0} \right) \right] + \frac{N^* \gamma^* c^2 R}{2R_0} I_1 \left( \frac{R}{2R_0} \right) K_1 \left( \frac{R}{2R_0} \right) + \frac{\gamma_0 c^2}{2} \\ &\rightarrow \frac{N^* \beta^* c^2}{R^2} + \frac{N^* \gamma^* c^2}{2} + \frac{\gamma_0 c^2}{2} \end{aligned} \quad (166)$$

$$\beta^* = 1.48 \times 10^5 \text{ cm}, \quad \gamma^* = 5.42 \times 10^{-41} \text{ cm}^{-1}, \quad \gamma_0 = 3.06 \times 10^{-30} \text{ cm}^{-1} . \quad (167)$$

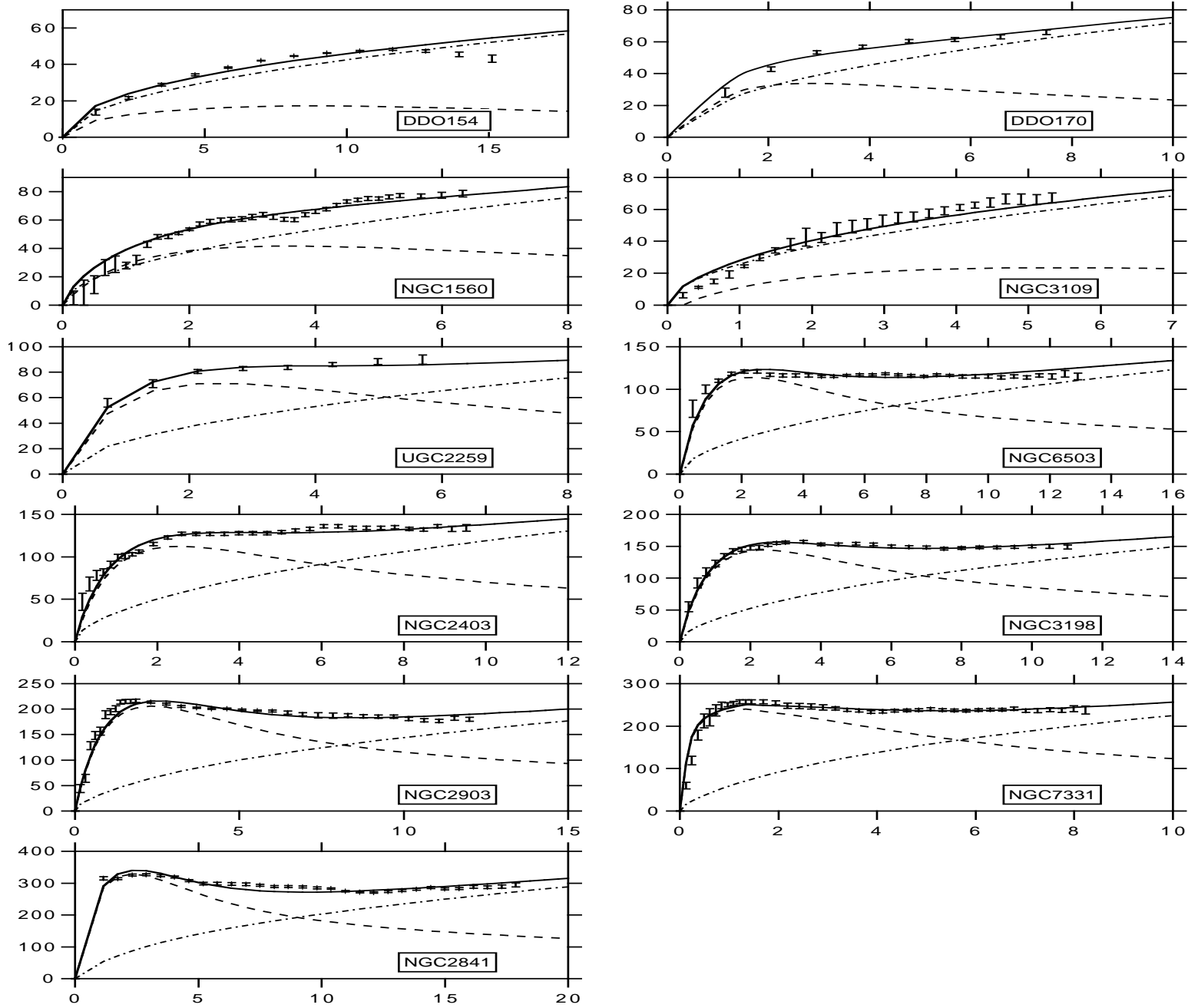


Figure 1: Some typical measured galactic rotation velocities

**Table 1: Characteristics of the eleven galaxy sample**

Galaxy	Distance (Mpc)	Luminosity ( $10^9 L_{B_\odot}$ )	$R_0$ (kpc)	$(v^2/c^2 R)_{\text{last}}$ ( $10^{-30} \text{cm}^{-1}$ )	$(M/L)$ ( $M_\odot L_{B_\odot}^{-1}$ )
DDO 154	3.80	0.05	0.48	1.51	0.71
DDO 170	12.01	0.16	1.28	1.63	5.36
NGC 1560	3.00	0.35	1.30	2.70	2.01
NGC 3109	1.70	0.81	1.55	1.98	0.01
UGC 2259	9.80	1.02	1.33	3.85	3.62
NGC 6503	5.94	4.80	1.73	2.14	3.00
NGC 2403	3.25	7.90	2.05	3.31	1.76
NGC 3198	9.36	9.00	2.72	2.67	4.78
NGC 2903	6.40	15.30	2.02	4.86	3.15
NGC 7331	14.90	54.00	4.48	5.51	3.03
NGC 2841	9.50	20.50	2.39	7.25	8.26

## 12 STANDARD COSMOLOGY

$$ds^2 = -c^2 dt^2 + R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \quad (168)$$

$$H = \frac{\dot{R}}{R}, \quad q = -\frac{R\ddot{R}}{\dot{R}^2} \quad (169)$$

$$T_{\mu\nu} = (\rho_M + p_M)U_\mu U_\nu + p_M g_{\mu\nu} - \Lambda g_{\mu\nu}, \quad \rho_M = \frac{A}{R^n} \quad (170)$$

$$\dot{R}^2 + kc^2 = \dot{R}^2 [\Omega_M + \Omega_\Lambda] \quad (171)$$

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = \left( \frac{n}{2} - 1 \right) \Omega_M - \Omega_\Lambda \quad (172)$$

$$\Omega_M = \frac{8\pi G \rho_M}{3c^2 H^2}, \quad \Omega_\Lambda = \frac{8\pi G \Lambda}{3cH^2}, \quad (173)$$

$$\frac{\Omega_\Lambda}{\Omega_M} = \frac{c\Lambda}{\rho_M} = \frac{T_V^4}{T^4} \quad (174)$$

### 13 CONFORMAL COSMOLOGY

$$I_M = - \int d^4x (-g)^{1/2} \left[ \frac{1}{2} S^{;\mu} S_{;\mu} - \frac{1}{12} S^2 R^\mu{}_\mu + \lambda S^4 + i \bar{\psi} \gamma^\mu [\partial_\mu + \Gamma_\mu] \psi - h S \bar{\psi} \psi \right] \quad (175)$$

$$T_{\mu\nu} = (\rho_M + p_M) U_\mu U_\nu + p_M g_{\mu\nu} - \frac{1}{6} S_0^2 \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha{}_\alpha \right) - g_{\mu\nu} \lambda S_0^4 \quad (176)$$

$$C_{\lambda\mu\nu\kappa} = 0, \quad T_{\mu\nu} = 0 \quad (177)$$

$$\frac{1}{6} S_0^2 \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha{}_\alpha \right) = (\rho_M + p_M) U_\mu U_\nu + p_M g_{\mu\nu} - g_{\mu\nu} \lambda S_0^4 \quad (178)$$

$$G_{\text{eff}} = -\frac{3}{4\pi S_0^2} \quad (179)$$

$$\dot{R}^2 + kc^2 = \dot{R}^2 [\bar{\Omega}_M + \bar{\Omega}_\Lambda], \quad q = -\frac{R\ddot{R}}{\dot{R}^2} = \left(\frac{n}{2} - 1\right) \bar{\Omega}_M - \bar{\Omega}_\Lambda \quad (180)$$

$$\bar{\Omega}_M = \frac{8\pi G_{\text{eff}} \rho_M}{3c^2 H^2}, \quad \bar{\Omega}_\Lambda = \frac{8\pi G_{\text{eff}} \Lambda}{3cH^2}, \quad \frac{\bar{\Omega}_\Lambda}{\bar{\Omega}_M} = \frac{c\Lambda}{\rho_M} = -\frac{T_V^4}{T^4} \quad (181)$$

$$\bar{\Omega}_\Lambda = \left(1 - \frac{T^2}{T_{\text{max}}^2}\right)^{-1} \left(1 + \frac{T^2 T_{\text{max}}^2}{T_V^4}\right)^{-1}, \quad \bar{\Omega}_M = -\frac{T^4}{T_V^4} \bar{\Omega}_\Lambda \quad (182)$$

$$0 \leq \bar{\Omega}_\Lambda \leq 1, \quad -1 \leq q \leq 0 \quad (183)$$

$$d_L = -\frac{c}{H_0} \frac{(1+z)^2}{q_0} \left(1 - \left[1 + q_0 - \frac{q_0}{(1+z^2)}\right]^{1/2}\right) \quad (184)$$

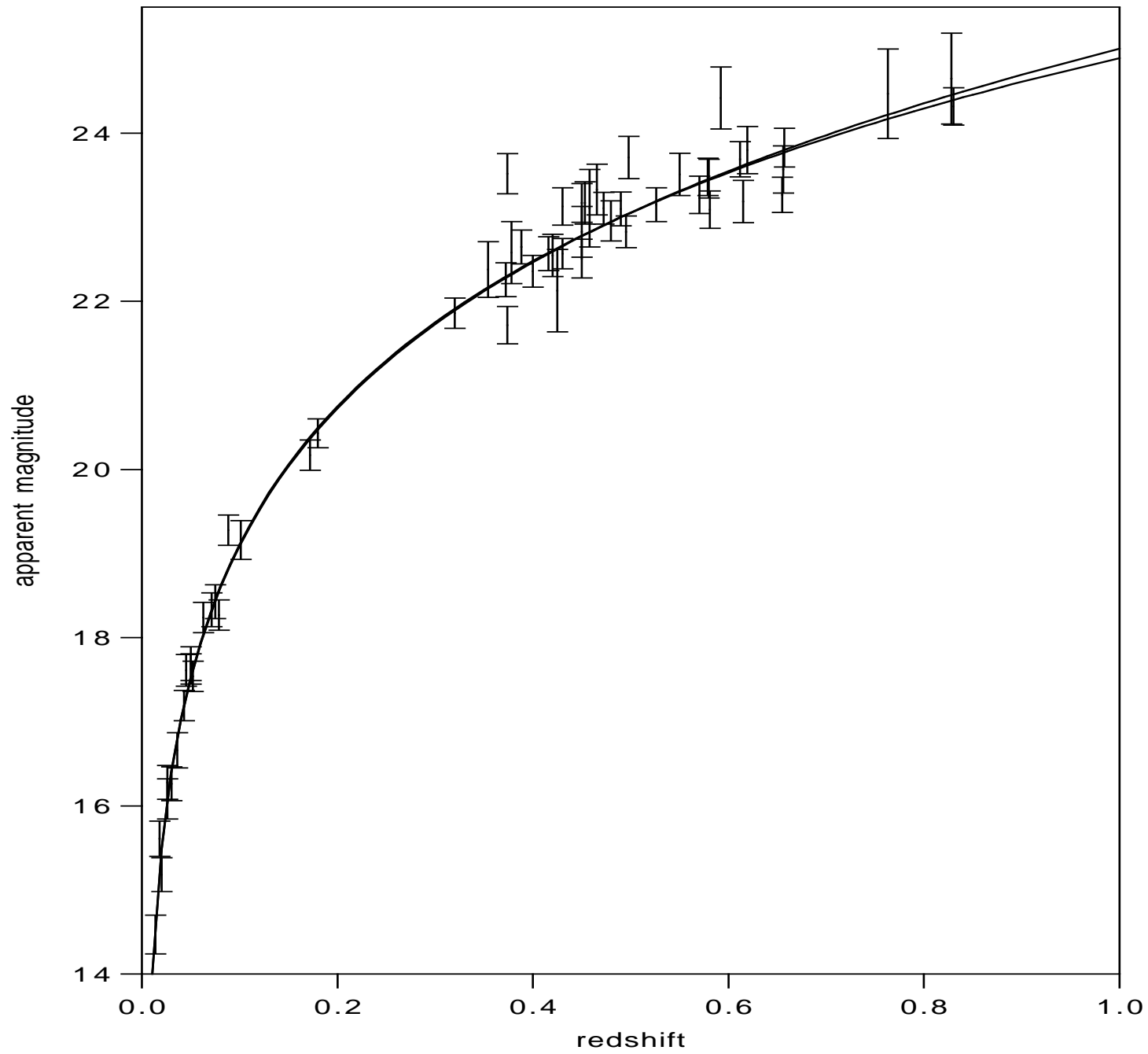


Figure 2: The  $q_0 = -0.37$  conformal gravity fit (upper curve) and the  $\Omega_M(t_0) = 0.3$ ,  $\Omega_\Lambda(t_0) = 0.7$  standard model fit (lower curve) to the  $z < 1$  supernovae Hubble plot data.

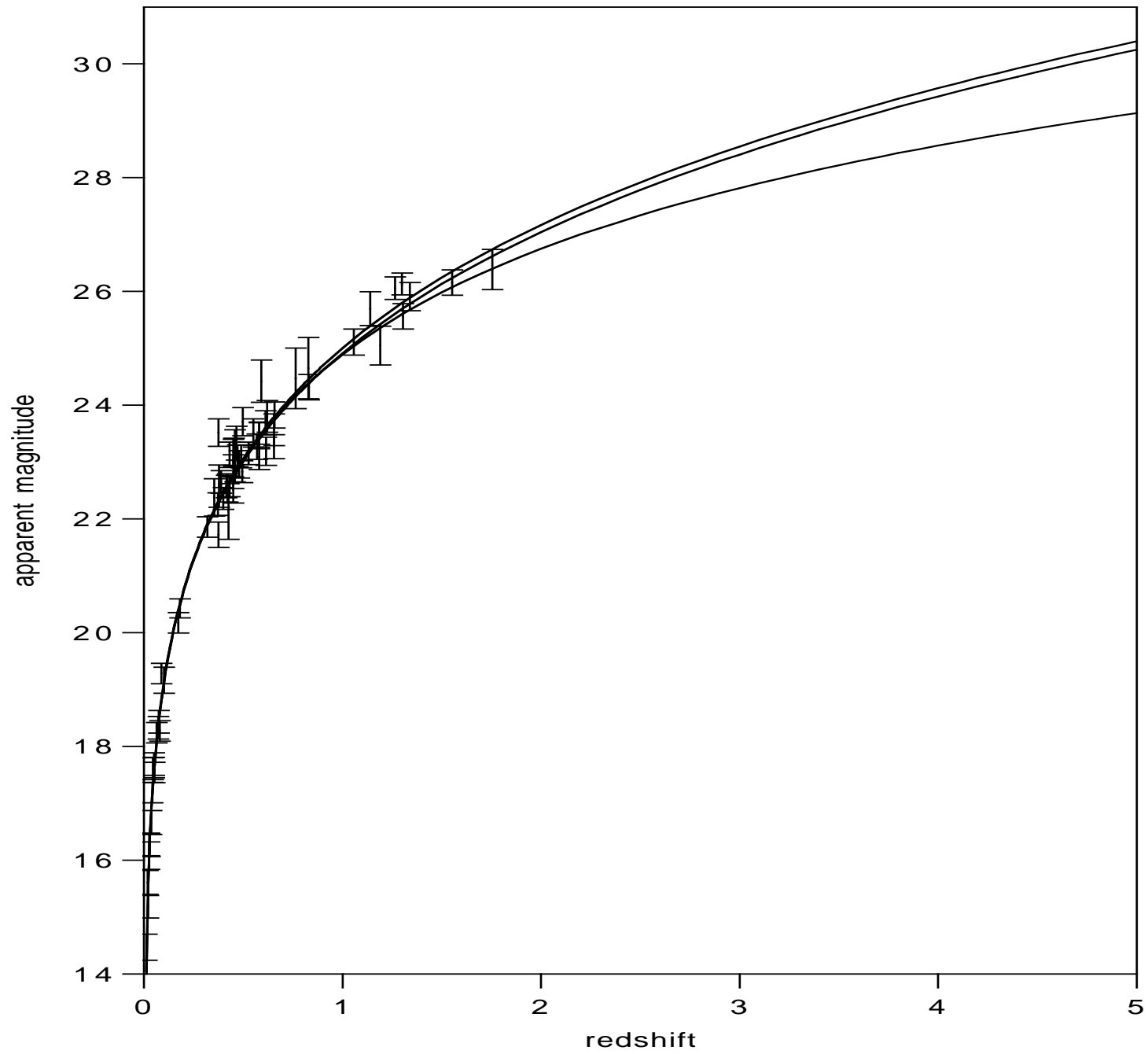


Figure 3: Hubble plot expectations for  $q_0 = -0.37$  (highest curve) and  $q_0 = 0$  (middle curve) conformal gravity and for  $\Omega_M(t_0) = 0.3$ ,  $\Omega_\Lambda(t_0) = 0.7$  standard gravity (lowest curve).

# CONFORMAL GRAVITY CHALLENGES STRING THEORY – UNITARITY OF FOURTH ORDER DERIVATIVE THEORIES

Philip D. Mannheim, Presentation at ICHEP-08, July 2008

## GHOST PROBLEM AND UNITARITY

1. C. M. Bender and P. D. Mannheim, *No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model*, June 2007 (0706.0207 [hep-th]). Phys. Rev. Lett. **100**, 110402 (2008).
2. P. D. Mannheim, *Conformal Gravity Challenges String Theory*, Pascos-07, July 2007 (0707.2283 [hep-th]).
3. C. M. Bender and P. D. Mannheim, *Giving up the ghost*, October 2007 (0807.2607 [hep-th]). Jour. Phys. A **41**, 304018 (2008)
4. C. M. Bender and P. D. Mannheim, *Exactly solvable PT-symmetric Hamiltonian having no Hermitian counterpart*, April 2008 (0804.4190 [hep-th]). Phys. Rev. D **78**, 025022 (2008).

## PT QUANTUM MECHANICS

5. C. M. Bender, *Making Sense of Non-Hermitian Hamiltonians*, March 2007 (hep-th/070309). Rep. Prog. Phys. **70**, 947 (2007).

## PAIS-UHLENBECK FOURTH ORDER OSCILLATOR

6. P. D. Mannheim and A. Davidson, *Fourth order theories without ghosts*, January 2000 (hep-th/0001115).
7. P. D. Mannheim and A. Davidson, *Dirac quantization of the Pais-Uhlenbeck fourth order oscillator*, August 2004 (hep-th/0408104). Phys. Rev. A **71**, 042110 (2005).
8. P. D. Mannheim, *Solution to the ghost problem in fourth order derivative theories*, August 2006 (hep-th/0608154). Found. Phys. **37**, 532 (2007).

## CONFORMAL GRAVITY AND THE COSMOLOGICAL CONSTANT PROBLEM

9. P. D. Mannheim, *Alternatives to dark matter and dark energy*, May 2005 (astro-ph/0505266). Progress in Particle and Nuclear Physics **56**, 340 (2006).

## TWO BIG PROBLEMS: COSMOLOGICAL CONSTANT and QUANTUM GRAVITY SHOULD HAVE A COMMON SOLUTION

To solve them both at once need a theory of gravity with dimensionless coupling constants and no fundamental mass scales. Then theory is **renormalizable** and there is **no cosmological constant at the level of the Lagrangian**. To achieve impose invariance under local conformal transformations of the form

$$g_{\mu\nu}(x) \rightarrow e^{2\alpha(x)} g_{\mu\nu}(x), \quad (185)$$

as it leads to a unique gravitational action

$$\begin{aligned} I_W &= -\alpha_g \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \\ &= -2\alpha_g \int d^4x (-g)^{1/2} \left[ R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} (R^\alpha{}_\alpha)^2 \right] \end{aligned} \quad (186)$$

where  $\alpha_g$  is dimensionless and  $C_{\lambda\mu\nu\kappa}$  is the conformal Weyl tensor. The associated gravitational equations of motion are the **fourth-order derivative**:

$$4\alpha_g [2C^{\mu\lambda\nu\kappa}{}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}] = T^{\mu\nu} \quad (187)$$

In the theory  $G_N$  is induced dynamically (just like the weak interaction  $G_F$ ), with this  $G_N$  only controlling local physics, and with the standard solar system predictions remaining intact. Because of early universe phase transitions an electroweak scale  $\Lambda$  is induced dynamically as the universe cools. However, at the same time an effective global cosmological  $G_{\text{eff}}$  is also induced, with this  $G_{\text{eff}}$  being found to naturally be altogether smaller than  $G_N$ . In the theory the standard  $\Omega_\Lambda = 8\pi G_N \Lambda / 3cH^2$  is replaced by  $\bar{\Omega}_\Lambda = 8\pi G_{\text{eff}} \Lambda / 3cH^2$ , and one can prove a bound

$$0 \leq \bar{\Omega}_\Lambda \leq 1, \quad -1 \leq q_0 \leq 0 \quad (188)$$

no matter how big  $\Lambda$  might be. The contribution of  $\Lambda$  to cosmic evolution is thus naturally tamed without any fine-tuning. **In conformal gravity then one does not need to quench  $\Lambda$  at all. Rather, one quenches the amount by which  $\Lambda$  gravitates.**

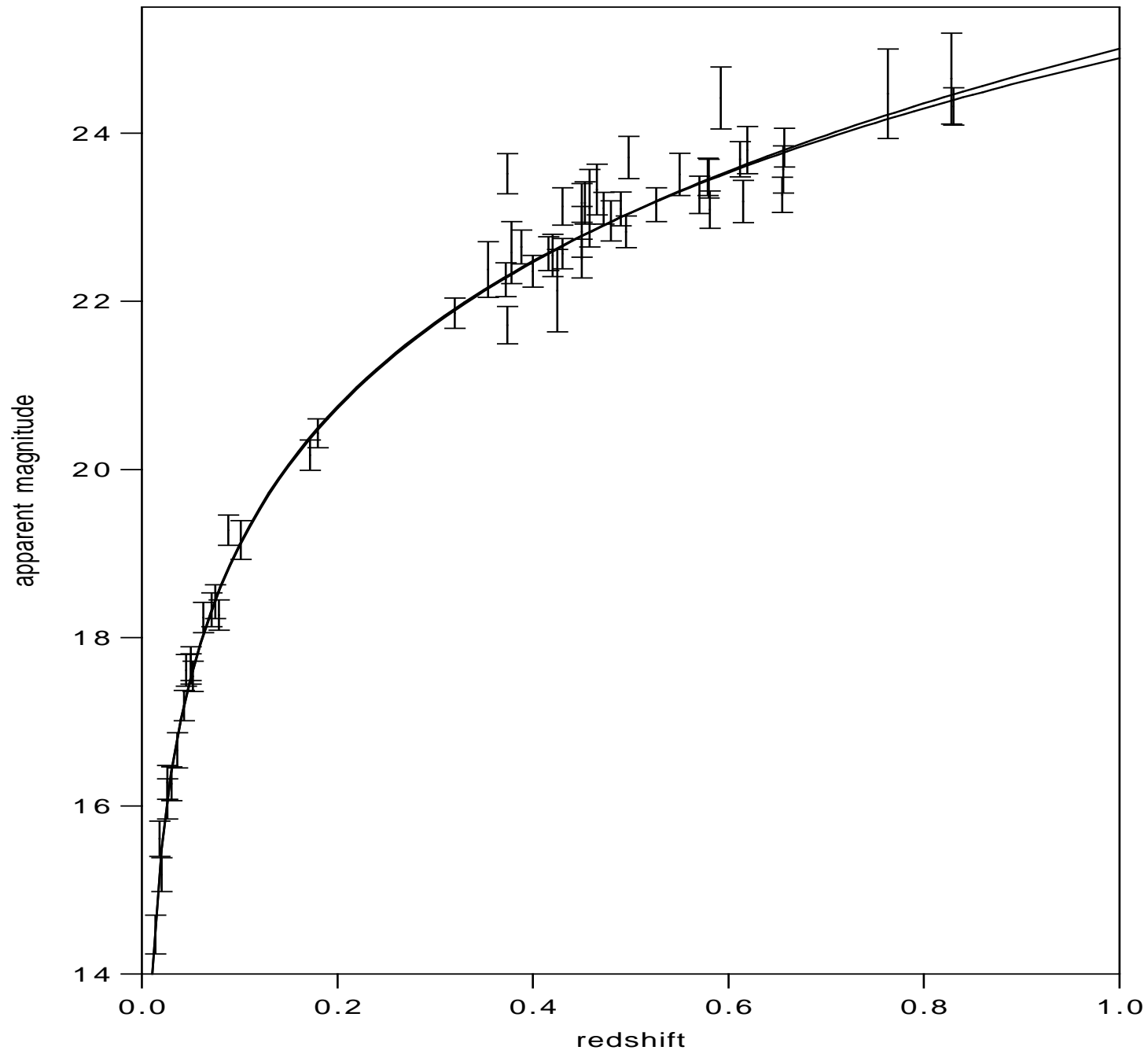


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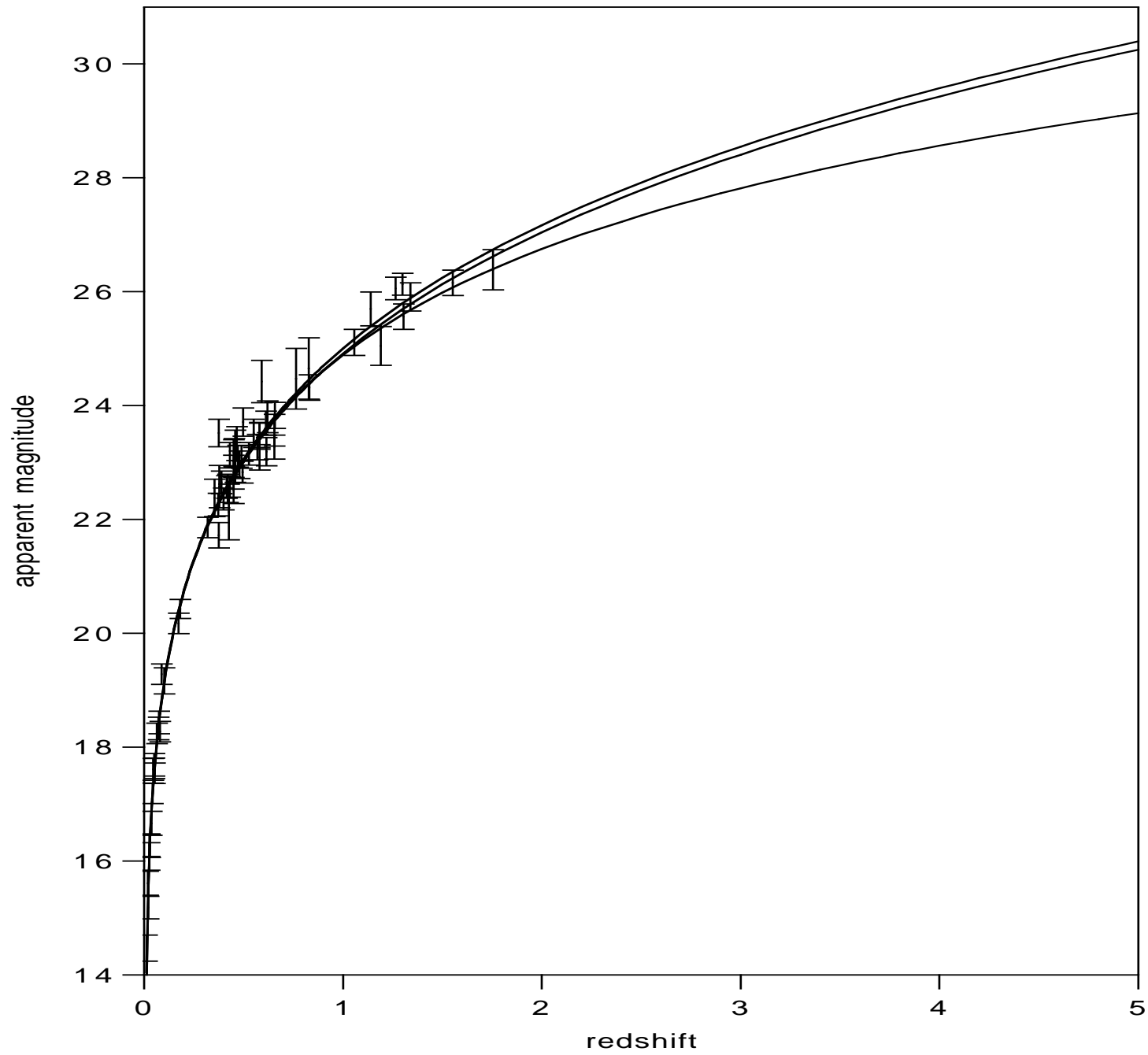


Figure 5: Hubble plot expectations for  $q_0 = -0.37$  (highest curve) and  $q_0 = 0$  (middle curve) conformal gravity and for  $\Omega_M(t_0) = 0.3$ ,  $\Omega_\Lambda(t_0) = 0.7$  standard gravity (lowest curve).