Ternary algebras ... some simple examples

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based on work with D B Fairlie and C K Zachos
and with X Jin and L Mezincescu

Ternary algebras seem to be gaining in their usefulness and importance to physics. For example, recently there has been progress in constructing a world-volume Lagrangian description for M2-branes. (cf. previous talks by Gomis, Nilsson, and Bergshoeff)

Ternary algebras are built from operations on three things at a time – so-called *3-brackets.*

Three specific realizations of 3-brackets

Jacobian or "classical" (Nambu 1973; Filippov 1985) acting on functions of x, y, z.

$$\{A, B, C\} = \frac{\partial (A, B, C)}{\partial (x, y, z)}$$

Operator or "quantum" (Nambu 1973) composed of sum of trinomials.

$$[A, B, C] = ABC - BAC + CAB - ACB + BCA - CBA$$

Commutator $\times$ trace (Awata, Li, Minic, and Yoneya 1999)

$$\langle A, B, C\rangle = [A, B]\ \ Tr\,(C) + [C, A]\ \ Tr\,(B) + [B, C]\ \ Tr\,(A)$$

3 Properties

(1) Linear and antisymmetric.

(2) Or rather, *2 out of 3* satisfy the Filippov condition (FI)

$$\{a, b, \{C, D, E\}\} = \{\{a, b, C\}, D, E\} + \{C, \{a, b, D\}, E\} + \{C, D, \{a, b, E\}\}$$

$$\langle a, b, \langle C, D, E\rangle\rangle = \langle\langle a, b, C\rangle, D, E\rangle + \langle C, \langle a, b, D\rangle, E\rangle + \langle C, D, \langle a, b, E\rangle\rangle$$

But in general this does ***not*** hold for associative operator products

$$0 \neq [a, b, [C, D, E]] - [[a, b, C], D, E] - [C, [a, b, D], E] - [C, D, [a, b, E]]$$

$$\equiv \mathrm{fi}\,(a, b; C, D, E) \qquad \text{the “Filippovian”}$$

The condition $\mathrm{fi}\,(a, b; C, D, E) = 0$ is ***not*** an operator identity.

In this sense it differs from the Jacobi identity for associative products. A *2-bracket-acting-on-2-bracket* situation.

$$[a, [b, C]] = \frac{1}{2}[C, [a, b]]$$

where lower case entries are antisymmetrized (signed sum over all perms).

(3) Associative operator **identity** (Bremner 1998; Nuyts 2008)

$$[[A,[b,c,d]\,,e]\,,f,g]=[[A,b,c]\,,[d,e,f]\,,g]$$

where lower case entries are again totally antisymmetrized. A *3-on-3-on-3* situation.

This *is* a consequence of associativity. If you posit an operator 3-bracket *based on associative products*, and it does not satisfy this, then you have *erred*. This is a *necessary* condition to realize a ternary algebra in terms of operator brackets.

Remark: The classical bracket and the ALMY bracket both satisfy the Bremner identity.

Remark: An operator 3-bracket acting on another 3-bracket does *not* close to give another 3-bracket, but rather a *5-bracket*.

$$[A,B,[C,D,E]]\propto[A,B,C,D,E]$$

Again, total antisymmetrization of the LHS is understood.

4 Examples

(1) Nambu $su(2)$.

$$[L_x, L_y, L_z] \equiv L_x [L_y, L_z] + L_y [L_z, L_x] + L_z [L_x, L_y]$$

$$= i \left(L_x^2 + L_y^2 + L_z^2 \right)$$

So, rescale by a fourth root of the Casimir

$$Q_x = \frac{L_x}{\sqrt[4]{L^2}} \ , \quad Q_y = \frac{L_y}{\sqrt[4]{L^2}} \ , \quad Q_z = \frac{L_z}{\sqrt[4]{L^2}} \ ,$$

and define a fourth charge as that fourth root,

$$Q_t = \sqrt[4]{L^2}$$

Then

$$[Q_a, Q_b, Q_c] = i \ \varepsilon_{abcd} \ Q^d$$

where $\varepsilon_{xyzt} = +1$ with a $\lceil -1, -1, -1, +1 \rfloor$ Lorentz signature.

The usual $\varepsilon\varepsilon$ identities imply this example is special: The Filippov condition holds for Nambu $su\,(2)$.

Nambu $su(2)$ has sub-algebras that close under 3-brackets

$$Q_x\ , \quad Q_y\ , \quad Q_z \pm Q_t$$

These can be realized in terms of the classical 3-bracket. Consider

$$x\sqrt{z}\ , \quad y\sqrt{z}\ , \quad z$$

$$x\sqrt{z}\ , \quad y\sqrt{z}\ , \quad x^2 + y^2$$

(2) Oscillator.

The usual four charges 1 , a , $a^\dagger$, and $N = a^\dagger a$ give

$$[1, N, a] = -a \ , \quad \left[1, N, a^\dagger\right] = a^\dagger \ , \quad \left[1, a, a^\dagger\right] = 1 \ , \quad \left[N, a, a^\dagger\right] = -1 - N$$

Three of these reduce to commutators: $[1, N, a] = [N, a]$, $\left[1, N, a^\dagger\right] = \left[N, a^\dagger\right]$, and $\left[1, a, a^\dagger\right] = \left[a, a^\dagger\right]$.

Alternatively,

$$R_1 = N \ , \quad R_2 = \frac{1}{\sqrt{2}} \left(a^\dagger + a\right) \ , \quad R_3 = \frac{1}{\sqrt{2}} i \left(a^\dagger - a\right) \ , \quad R_4 = N + 1$$

Then we are back to Nambu $su\,(2)$ more or less (actually, $sl\,(2, \mathbb{R})$)

$$[R_a, R_b, R_c] = i\epsilon_{abcd} \ R^d$$

with $\epsilon_{1234} = +1$, again with Lorentz metric to raise indices, $\eta_{ab} = \lceil 1, 1, 1, -1 \rfloor$.

So, what's new here? Two additional bilinears, a^2 and $a^{\dagger 2}$. Three brackets of these give trilinears.

$$\left[a, a^2, a^{\dagger 2}\right] = 2a + 2Na \ , \quad \left[a^\dagger, N, a^2\right] = -2a - Na \ , \quad \left[a, N, a^2\right] = -a^3 \ ,$$

$$\left[a^\dagger, a^2, a^{\dagger 2}\right] = 2a^\dagger + 2a^\dagger N \ , \quad \left[a, N, a^{\dagger 2}\right] = 2a^\dagger + a^\dagger N \ , \quad \left[a^\dagger, N, a^{\dagger 2}\right] = a^{\dagger 3} \ .$$

Etc. Hence the ternary algebra becomes *infinite*. The full enveloping algebra for the oscillator.

The oscillator enveloping algebra does *not* satisfy Filippov's condition. For example

$$-2 = \left[\left[a^\dagger, a^\dagger a, a^{\dagger 2}\right], a, a^2\right] - \left[\left[a^\dagger, a, a^2\right], a^\dagger a, a^{\dagger 2}\right] - \left[a^\dagger, \left[a^\dagger a, a, a^2\right], a^{\dagger 2}\right] - \left[a^\dagger, a^\dagger a, \left[a^{\dagger 2}, a, a^2\right]\right]$$

$$20a^\dagger = \left[\left[a^\dagger a, a^{\dagger 2}, a^2\right], a, a^{\dagger 2}\right] - \left[\left[a^\dagger a, a, a^{\dagger 2}\right], a^{\dagger 2}, a^2\right] - \left[a^\dagger a, \left[a^{\dagger 2}, a, a^{\dagger 2}\right], a^2\right] - \left[a^\dagger a, a^{\dagger 2}, \left[a^2, a, a^{\dagger 2}\right]\right]$$

But of course it *does* satisfy the Bremner identity.

(3) Infinite Virasoro–Witt 3-algebra.

An infinite, closed sub-algebra of the oscillator algebra.

$$L_n = -\left(a^{\dagger}\right)^n N$$

for $n \geq 0$. The commutator algebra is

$$[L_n, L_m] = (n - m)\, L_{n+m}$$

for $m, n \geq 0$.

The corresponding Nambu 3-brackets are

$$[L_n, L_m, L_k] = 0$$

Thus we have a *null 3-algebra* for an infinite set of non-trivial, non-commuting oscillator charges.

The Filippov condition is trivially satisfied in this case.

And of course, the Bremner identity works.

More generally, we may modify the oscillator realization to

$$L_n = -\left(a^\dagger\right)^n (N + \gamma + n\beta) \ , \quad [L_n, L_m] = (n-m)\, L_{n+m}$$

The parameter β is related to the $sl\,(2,\mathbb{R})$ Casimir, $C = \beta\,(1-\beta)$.

Now we find a non-null 3-bracket when $0 \neq \beta \neq 1$.

$$[L_n, L_m, L_k] = \beta\,(1-\beta)\ \ (n-m)\,(m-k)\,(n-k)\ \ \left(a^\dagger\right)^{n+m+k}$$

The Filippov condition still holds for these modified Ls.

$$[L_r, L_s, [L_k, L_m, L_n]] = [L_m, L_n, [L_k, L_r, L_s]] + [L_k, L_n, [L_r, L_m, L_s]] + [L_k, L_m, [L_r, L_s, L_n]]$$

But to close the 3-algebra, we must consider all additional 3-brackets involving powers of $a^\dagger$.

$$\left[\left(a^\dagger\right)^n, L_m, L_k\right] = (m-k)\left(L_{n+m+k} + (1-2\beta)\, n \left(a^\dagger\right)^{n+m+k}\right)$$

$$\left[\left(a^\dagger\right)^n, \left(a^\dagger\right)^m, L_k\right] = (m-n)\left(a^\dagger\right)^{n+m+k}$$

$$\left[\left(a^\dagger\right)^n, \left(a^\dagger\right)^m, \left(a^\dagger\right)^k\right] = 0$$

We must then check all the Filippov conditions involving these new 3-brackets.

First rewrite as

$$Q_k \equiv \frac{1}{\sqrt[4]{\beta\,(1-\beta)}} L_k \ , \quad R_k \equiv \sqrt[4]{\beta\,(1-\beta)}\,\left(a^{\dagger}\right)^k$$

then

$$[Q_k, Q_m, Q_n] = (k-m)\,(m-n)\,(k-n)\ \ R_{k+m+n}$$

$$[Q_p, Q_q, R_k] = (p-q)\ \ (Q_{k+p+q} + z\ k\ R_{k+p+q})$$

$$[Q_p, R_q, R_k] = (k-q)\ \ R_{k+p+q}$$

$$[R_p, R_q, R_k] = 0$$

where $z \equiv (1-2\beta)\,/\sqrt{\beta\,(1-\beta)}$.

The Bremner identities hold for this ternary algebra. So the algebra is consistent with an underlying associative operator product no matter how it is realized.

The Filippov conditions now *fail* when only one R is involved.

There are two such cases out of twelve possibilities. In an obvious notation, the twelve Filippov possibilities stem from each of the following:

$$[R,R,[R,R,R]]\ ,\quad [R,R,[Q,R,R]]\ ,\quad [Q,R,[R,R,R]]\ ,\quad [R,R,[Q,Q,R]]\ ,\quad [Q,R,[Q,R,R]]\ ,\quad [Q,Q,[R,R,R]]\ ,$$

$$[Q,Q,[Q,Q,Q]]\ ,\quad [Q,Q,[R,Q,Q]]\ ,\quad [R,Q,[Q,Q,Q]]\ ,\quad [Q,Q,[R,R,Q]]\ ,\quad [R,Q,[R,Q,Q]]\ ,\quad [R,R,[Q,Q,Q]]\ .$$

The two exceptions, which for generic z do *not* obey FI conditions, give instead

$$[Q_p,Q_q,[Q_k,Q_m,R_n]]-[[Q_p,Q_q,Q_k],Q_m,R_n]-[Q_k,[Q_p,Q_q,Q_m],R_n]-[Q_k,Q_m,[Q_p,Q_q,R_n]]$$

$$=\left(4+z^2\right)\ (p-q)(k-m)(m-p-q+k)\,n\ R_{k+m+n+p+q}\ ,$$

$$[Q_p,R_q,[Q_k,Q_m,Q_n]]-[[Q_p,R_q,Q_k],Q_m,Q_n]-[Q_k,[Q_p,R_q,Q_m],Q_n]-[Q_k,Q_m,[Q_p,R_q,Q_n]]$$

$$=\left(4+z^2\right)\ (n-k)(k-m)(m-n)\,q\ R_{k+m+n+p+q}\ .$$

Nevertheless, for the special cases $z=\pm 2i$ the RHSs here also vanish.

It is interesting that the special values $z = \pm 2i$ are obtained in this realization only in the "classical" limit of large $sl\,(2, \mathbb{R})$ Casimirs, $C = \beta\,(1-\beta) \to -\infty$, for which

$$z^2 = \frac{(1-2\beta)^2}{\beta\,(1-\beta)} \underset{\beta \to \pm\infty}{\longrightarrow} -4\ .$$

This is effectively a contraction of the original operator algebra.

(4) Infinite classical bracket algebra for exponentials

$$E_a = \exp(a \cdot r)$$

This leads to

$$\{E_a, E_b, E_c\} = a \cdot (b \times c) \;\; E_{a+b+c}$$

The indices here are 3-vectors, with $\cdot$ and $\times$ the usual dot and cross products.

This infinite algebra *does* satisfy Filippov's condition, since all classical brackets do, as well as the Bremner identity.

Not known: Realization as operator brackets.

But, it *is* known how to realize the classical limit of the Virasoro-Witt ternary operator algebra in terms of classical 3-brackets. Consider

$$L_n = (x - c_n y)\, e^{nz} \ , \quad M_n = y e^{nz}$$

where $M_k \sim \left(a^{\dagger}\right)^k$. With the choice $c_k = (\alpha - \beta k)$, we obtain

$$\frac{\partial\,(L_k, L_m, L_n)}{\partial\,(x, y, z)} = -\beta^2\,(k-m)\,(k-n)\,(m-n)\,M_{k+m+n}$$

$$\frac{\partial\,(L_k, L_m, M_n)}{\partial\,(x, y, z)} = (k-m)\,(L_{k+m+n} - 2\beta n M_{k+m+n})$$

$$\frac{\partial\,(L_k, M_m, M_n)}{\partial\,(x, y, z)} = (n-m)\,M_{k+m+n}$$

$$\frac{\partial\,(M_k, M_m, M_n)}{\partial\,(x, y, z)} = 0$$

This differs from the original operator Virasoro-Witt ternary algebra only in the β-dependent coefficients on the RHS. Namely, $-\beta^2$ appears instead of $\beta\,(1-\beta)$ and -2β instead of $1-2\beta$. So, we may identify this classical 3-algebra as a realization of the infinite Casimir limit, $\beta \to \pm\infty$, of the quantum algebra.

It is of interest to include central charges, say through the use of operator product expansions, and to investigate ternary algebras in the context of CFT. This is work in progress.

Thank you.

Time to get ready for cocktails and the banquet!

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