How Galilean invariant theories like Navier-Stokes are similar to gauge theories

Arjun Berera

The University of Edinburgh

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Overview

- Navier-Stokes equation (NSE) and Galilean invariance
- Standard path integral representation of NSE
- Problems with standard path integral and how to correct them
- Relations between correlation functions - implications for renormalization of vertex function
- Extended Galilean Invariance

Stochastic Navier-Stokes equation

$$\frac{\partial V_m(\mathbf{x},t)}{\partial t} + \lambda_0 P_{ml}(\nabla)\frac{\partial (V_n(\mathbf{x},t)V_l(\mathbf{x},t))}{\partial x_n} - \nu_0 \nabla^2 V_m(\mathbf{x},t) = \eta_m(\mathbf{x},t)$$

Invariant under Galilean transformations:

$$\begin{aligned}
\mathbf{V}(\mathbf{x},\mathbf{t}) &= \mathbf{V}'(\mathbf{x}',t') + \mathbf{c} \\
\sigma(\mathbf{x},\mathbf{t}) &= \sigma'(\mathbf{x}',\mathbf{t}') \\
t &= t' \\
\mathbf{x} &= \mathbf{x}' + \mathbf{c}t.
\end{aligned}$$

NSE in momentum-frequency space

Fourier transform:

$$V_m(\mathbf{k},\omega) = (2\pi)^{-4} \int d^3x dt V_m(\mathbf{x},t)\, \exp(i\mathbf{k}\cdot\mathbf{x} - \omega t)$$

Stochastic Navier-Stokes equation in momentum-frequecy space:

$$(-i\omega + \nu k^2)V_m(\mathbf{k},\omega) - M_{mnl}(\mathbf{k}) \int d\mathbf{j} d\omega_1 V_n(\mathbf{k}-\mathbf{j},\omega-\omega_1)V_l(\mathbf{j}\,,\omega_1) = \eta_m(\mathbf{k},\omega)$$

Galilean invariance in momentum space:

$$\begin{aligned} \mathbf{k}' &= \mathbf{k} \\ \omega' &= \omega - \mathbf{c}.\mathbf{k} \\ \mathbf{V}(\mathbf{k},\omega) &= \mathbf{V}'(\mathbf{k},\omega-\mathbf{c}.\mathbf{k}) + \mathbf{c}\,\delta^3(\mathbf{k})\delta(\omega) \\ \sigma(\mathbf{k},\omega) &= \sigma'(\mathbf{k},\omega-\mathbf{c}.\mathbf{k}). \end{aligned}$$

One is interested in computing correlation functions such as $\langle V_i(\mathbf{k},\omega)V_j(\mathbf{k}',\omega')\rangle$, etc...

Path integral for NSE

(Martin, Siggia, Rose, Phys. Rev. A **8**, 423 (1973); R. D. Jensen, J. Stat. Phys. **25**, 183 (1981))

Generating functional or path integral for NSE theory:

$$Z = \int [D\mathbf{V}][D\sigma] \exp\{-S[\mathbf{V},\sigma] + \int d\mathbf{k} d\omega (\mathbf{J}\cdot\mathbf{V} + \mathbf{\Sigma}\cdot\sigma)\},$$

where the action

$$S[\mathbf{V},\sigma] = \frac{1}{2}\int d\mathbf{k} d\omega\ \sigma_i(-\mathbf{k},-\omega) D_{ij}(-\mathbf{k})\sigma_j(\mathbf{k},\omega) -$$
$$i\int d\mathbf{k} d\omega\ \sigma_\alpha(-\mathbf{k},-\omega)\Big[(-i\omega+\nu k^2)V_\alpha(\mathbf{k},\omega) -$$
$$M_{\alpha\beta\gamma}(\mathbf{k})\int d\mathbf{j} d\omega_1 V_\beta(\mathbf{k}-\mathbf{j},\omega-\omega_1)V_\gamma(\mathbf{j},\omega_1)\Big].$$

This is just like particle physics, but no knowledge of that is needed, this is just a multi-dimensional integral...

Can compute all correlation functions with Z by taking appropriate derivatives with respect to the "sources" J and Σ. For example

$$\langle V_i(\mathbf{k},\omega)V_j(\mathbf{k}',\omega')\rangle = \frac{\delta}{\delta \mathbf{J}_i(\mathbf{k},\omega)}\frac{\delta}{\delta \mathbf{J}_j(\mathbf{k}',\omega')} Z[J,\Sigma] \qquad \text{etc...}$$

Representing NSE in a path integral

$\mathbf{V_{soln}}$ denotes solution to NSE

Functional or path-integral representation for an arbitrary function F (trivial identity)

$$F(\mathbf{V}_{soln}) = \int [\mathcal{D}\mathbf{V}]\, F(\mathbf{V})\, \delta[\mathbf{V} - \mathbf{V}_{soln}], \tag{1}$$

Solutions of NSE equations depends on the random noise sources: that is, $\mathbf{V}_{soln} = \mathbf{V}_{soln}[\eta]$

Solutions of NSE averaged over the random force:

$$\langle F(\mathbf{V})\rangle = \int [\mathcal{D}\eta]\, F(\mathbf{V}_{soln})\mathcal{P}[\eta],$$

Noise distribution functional $\mathcal{P}$ is assumed to be given and is normalized to unity

$$\int [\mathcal{D}\eta]\mathcal{P}[\eta] = 1.$$

re-express delta functional in (1) as follows:

$$\delta[\mathbf{V} - \mathbf{V}_{soln}] \;=\; \mathcal{J}\delta[\partial_t V_j + \lambda_0 \mathbf{P}_{jn}\partial_l(V_l V_n) - \nu\nabla^2 V_j - (\eta)_j] \tag{2}$$

Representing NSE in a path integral - cont.

Jacobian functional determinant $\mathcal{J}$:

$$(\partial_t + \nu\nabla^2)\delta_{jk} + \lambda_0 \mathbf{P}_{jk}\partial_l(V_l)\delta(\mathbf{x}-\mathbf{x}')\delta(t-t').$$

replace each of functional deltas appearing on the left-hand side of (2) and by its (functional) Fourier integral representation.

Recall for any (vector-valued) space-time field $\mathbf{\Phi}(\mathbf{x},\tau)$, we have

$$\delta[\mathbf{\Phi}] = \int [\mathcal{D}\vec{\sigma}]\,\mathsf{exp}\,(i\int d^d\vec{x}\,dt\,\vec{\sigma}\cdot\mathbf{\Phi}),$$

where $\vec{\sigma}$ is the Fourier conjugate field and the dot product indicates a sum over the repeated indices: $\vec{\sigma}\cdot\mathbf{\Phi} = \sigma_j\mathbf{\Phi}_j$, for $j = 1, 2, \cdots, d$.

Representing NSE in a path integral - cont.

Multiple functional integral representation for the stochastic average of an arbitrary function F of the exact solutions of NSE:

$$\begin{aligned}\langle F(\mathbf{V})\rangle \;&=\; \int [\mathcal{D}\eta][\mathcal{D}\sigma]\,[\mathcal{D}\mathbf{V}]\,F(\mathbf{V})\mathcal{J}\,\mathcal{P}[\sigma] \\ &\quad\times \exp\left(i\int d^d\vec{x}\,d\tau\;\sigma_j[\partial_t V_j + \lambda_0 \mathbf{P}_{jn}\partial_l(V_l V_n) - \nu\nabla^2 V_j - (\eta)_j]\right)\end{aligned}$$

convenient form for F.

$$F = \exp\left(\mathbf{J}\cdot\mathbf{V} + \boldsymbol{\Sigma}\cdot\sigma\right)$$

and we define the generating functional Z by

$$Z[\mathbf{J},\boldsymbol{\Sigma}] \equiv \langle \exp\left(\sigma\cdot\mathbf{J} + \boldsymbol{\Sigma}\cdot\sigma\right)\rangle. \qquad (3)$$

Path integral ill-defined

(AB, Hochberg, Phys. Rev. Lett. **99**, 254501 (2007))

The path integral of previous slide has been known for more than 2 decades, but it contains spurious relations

For example by performing a Galilean transformation on the path integral one finds:

$$\begin{aligned} & \langle \exp \int d\mathbf{k}d\omega\,\{\mathbf{J}(-)\cdot\mathbf{V}(\mathbf{k},\omega)+\Sigma(-)\cdot\sigma(\mathbf{k},\omega)\}\rangle \\ = & \langle \exp \int d\mathbf{k}d\omega\,\{\mathbf{J}(-)\cdot\mathbf{V}(\mathbf{k},\omega+\mathbf{c}\cdot\mathbf{k}) \\ - & \mathbf{c}\cdot\mathbf{J}(-)\delta^3(\mathbf{k})\delta(\omega)+\Sigma(-)\cdot\sigma(\mathbf{k},\omega+\mathbf{c}\cdot\mathbf{k})\}\rangle, \end{aligned} \tag{4}$$

The above equality leads to an infinite number of meaningless relations.

Spurious relations from standard part integral

If we apply $\delta/\delta J_i(-\mathbf{k},\omega)$ to both sides of Eq. (4), it gives:

$$\langle \mathbf{V}_i(\mathbf{k},\omega)\rangle = \langle \mathbf{V}_i(\mathbf{k},\omega+\mathbf{c}\cdot\mathbf{k}) - \mathbf{c}_i\delta^3(\mathbf{k})\delta(\omega)\rangle.$$

Integrating over an infinitesimal neighborhood near $\mathbf{k}=\mathbf{0}$, $\omega=0$, it leads to the relation $\mathbf{c}_i = 0$, which is meaningless.

Similarly, two derivative $\delta/\delta J_i(-\mathbf{k}_1,\omega_1)\delta/\delta J_j(-\mathbf{k}_2,\omega_2)$ leads to:

$$-c_j\Delta\mathbf{k}_1\Delta\omega_1\langle \mathbf{V}_i(\mathbf{0},0)\rangle - c_i\Delta\mathbf{k}_2\Delta\omega_2\langle \mathbf{V}_j(\mathbf{0},0)\rangle + c_ic_j = 0,$$

Meaningless!

Any number of further derivatives with respect to $\mathbf{J}$ and/or $\mathbf{\Sigma}$ will lead to an infinite number of meaningless relations.

Spurious relations from standard Ward identities

From generating functional $Z[\mathbf{J},\mathbf{\Sigma}]$ can also examine the standard Ward identities $\hat{O}_j Z = 0$, where

$$\hat{O}_j \equiv \int d\mathbf{k}d\omega\,(J_m(-)k_j\frac{\partial}{\partial\omega}\frac{\delta}{\delta J_m(-)} + \Sigma_m(-)k_j\frac{\partial}{\partial\omega}\frac{\delta}{\delta\Sigma_m(-)} - J_j(-)\delta(\mathbf{k})\delta(\omega)).$$

Eg. $\delta/\delta\mathbf{J}_i(-\mathbf{k},-\omega)\hat{O}_j|_{\mathbf{J}=\mathbf{\Sigma}=\mathbf{0}}Z = 0 \Rightarrow \langle k_j\frac{\partial}{\partial\omega}\mathbf{V}_i(\mathbf{k},\omega) - \delta_{ij}\delta^3(\mathbf{k})\delta(\omega)\rangle = 0$. Contract with $\delta_{ij} \Rightarrow \delta^3(\mathbf{k})\delta(\omega) = 0$. Meaningless.

Infinite number of spurious relations: Consider two derivatives $\delta/\delta\mathbf{J}_i(-\mathbf{k}_1,-\omega_1)\delta/\delta\mathbf{J}_m(-\mathbf{k},-\omega)\hat{O}_j|_{\mathbf{J}=\mathbf{\Sigma}=\mathbf{0}}Z = 0 \Rightarrow$ $\delta^3(\mathbf{k}_1)\delta(\omega_1)\langle\mathbf{V}_m(\mathbf{k},\omega)\rangle = 0$, Mean fluid velocity $\langle\mathbf{V}(\mathbf{k},\omega)\rangle$ can be generally nonzero for appropriate noise force. Meaningless relation.

One more derivative with respect to $\mathbf{J}$ or $\mathbf{\Sigma}$, irrespective of the properties of the noise force, resulting relation spurious

Similarly, additional number of such derivatives will lead to an infinite number of spurious relations, since they will wrongly imply higher order velocity correlation functions are vanishing.

Correcting the path integral

Observation: Since the path integral integrates over all values of the fields $\mathbf{V}$ and σ and for all modes, in particular included in this sum are field configurations related by just a overall velocity boost $\mathbf{c}$. But such field configurations are simply the *same* field configurations only observed in different inertial frames, i.e. related by Galilean transformations. Thus the path integral contains an infinite number of identical copies of the same field configuration which when summed over leads to an infinity, rendering the path integral to be ill-defined.

Solution to problem, choose just one inertial reference frame, so the zero mode of the velocity field is fixed to one value.

Fadeev Popov procedure

Our problem is analogous to "gauge invariance" in particle physics

Solved by (Fadeev and Popov, Phys. Lett. B **25**, 29 (1967))

Insert factor of unity $1 = \int d\mathbf{b}\, \delta^3(\mathbf{V}_0 - \mathbf{b})$ into path integral:

$$Z = \int [D\mathbf{V}][D\sigma] \int d\mathbf{b}\, \delta^3(\mathbf{V}_0 - \mathbf{b})\, \exp\{-S[\mathbf{V}, \sigma]\}.$$

Integrand is independent of $\mathbf{b}$ since different values of it can be related by Galilean transformation

$\Rightarrow$ the $d\mathbf{b}$ integral can completely factor out as an overall infinity as:

$$Z = \left(\int d\mathbf{b} \right) \int [D\mathbf{V}][D\sigma] \delta^3(\mathbf{V}_0 - \mathbf{b})\, \exp\{-S[\mathbf{V}, \sigma]\}.$$

Since path integral is independent of $\mathbf{b}$ can also choose any function of it to insert like $G(\mathbf{b}) = \exp(-\frac{1}{2\xi}\mathbf{b} \cdot \mathbf{b})$, where $\xi > 0$.

Leads to

$$Z'_{GF} = \int [D\mathbf{V}][D\sigma]\, \exp\{-S[\mathbf{V}, \sigma] - \frac{\mathbf{1}}{\mathbf{2\xi}} \mathbf{V}_0 \cdot \mathbf{V}_0\}.$$

Rediscovering the Galilean invariance

Path integral well defined - where is Galilean invariance of NSE?

The path integral contains a symmetry analogous to BRS symmetry of particle physics (Becchi, Rouet, Stora, Phys. Lett. B**52**, 344 (1974))

Multiply generating functional Z by constant: $\int d\eta d\eta^* \exp\{i\eta^* \cdot \eta\}$

η and η^* - constant complex conjugate Grassmann vectors:

$\{\eta_i, \eta_j^*\} = 0$, $\eta^2 = \eta^{*2} = 0$, independent of $\mathbf{k}$,ω.

"Gauge-fixed" action is:

$$S_{GF}[\mathbf{V}, \sigma, \eta, \eta^*] = S[\mathbf{V}, \sigma] + \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\xi}} \mathbf{V}_0 \cdot \mathbf{V}_0 - i\eta^* \cdot \eta \ ,$$

This action has "BRS" symmetry: In Galilean transformations defined earlier, replace velocity boost $\mathbf{c}$ by $\mathbf{c} \to c\zeta(\eta^* + \eta)$ and

$$\delta_{\mathrm{BRS}}\eta = -\frac{\boldsymbol{i}}{\boldsymbol{\xi}} \mathbf{V}_0 c\zeta, \qquad \delta_{\mathrm{BRS}}\eta^* = +\frac{\boldsymbol{i}}{\boldsymbol{\xi}} \mathbf{V}_0 c\zeta,$$

ζ is real Grassmann parameter

Galilean invariance is now restored

(similar to problem of abelian gauge theory)

Relations between correlation functions - Ward identities

If one performs the BRS transformation on our gauge fixed functional Z_{GF}, the action S_{GF} is BRS symmetric, but the source terms are not. This will lead to relations between correlation functions, typically called Ward identities in particle physics.

In particular, the following equation would emerge (AB, Hochberg, Phys. Rev. Lett. **99**, 254501 (2007)):

$$[\frac{i}{\xi}(\theta^* - \theta) \cdot \frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\mathbf{J_0}} + (\frac{\boldsymbol{\partial}}{\boldsymbol{\partial\theta_j}} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial\theta_j^*}})\hat{O}_j]Z_{GF} = 0,$$

Relating vertex and response functions

Interesting relation to emerge from NSE functional:

$$-k_m \frac{\partial}{\partial \omega} \Gamma^{(1,1)}_{ln}(-\mathbf{k}, -\omega; \mathbf{k}, \omega) = \Gamma^{(2,1)}_{mln}(\mathbf{0}, 0; -\mathbf{k}, -\Omega; \mathbf{k}, \omega). \quad (5)$$

Left-hand-side response function, right-hand-side vertex function

Note this relation arises in the limit of vanishing mean translational velocity but *independent* of properties of fluctuating field

(Consistent with kinematic arguments made earlier)

This result has implications for the renormalization of the nonlinear coupling term in the NSE

Renormalization of fluid system

How properties of fluid system change at different length scales

NSE is an effective description of a fluid, defined up to some short distance cut-off scale η, the viscosity scale, below which a kinetic theory desription of the individual fluid particles is needed.

One could ask what new equation describes the same system, except now the short distance cutoff scale is multiplied by a factor 2, and all physics between the scale η and 2η is now implicitly accounted for in the new equation.

The result in general could introduce new terms into the original equation, but in addition, the parameters in the original equation, here the viscosity and vertex coefficient, in general will also change, thus be renormalized.

The vertex renormalization issue

Varied views on consequences of Galilean invariance to the renormalization of the vertex:

- Forster, Nelson and Stephens, (Phys. Rev. A**16**, 732 (1977), in this classic paper state analysis is simplified by a Ward identity related to Galilean invariance and that vertex corrections vanish in the infrared limit.
- De Dominicis and Martin (Phys. Rev. A**19**, 419 (1979)) - state coupling (of nonlinear vertex term) suffers no renormalization.
- U. Frisch, (*Turbulence: the Legacy of A. N. Kolmogorov*, Cambridge U. Press, Cambridge, 1985) states in the FNS problem, the force, having a white-noise-dependence on the time, preserves the Galilean invariance of the NSE, so that vertex corrections are ruled out.

The vertex renormalization issue - cont.

- Mou and Weichman (Phys. Rev. E**52**, 3737 (1995)) derive Ward identities which are used to prove absence of ultraviolet divergences in the three-point vertex function of NSE.

- L. T. Adzhemyan, *et. al*, (*The Field Theoretic Renormalization Group in Fully Developed Turbulence*, Gordon and Breach, New York, 1999) - state Ward identities for Galilean transformation were first used (in DeDominicis and Martin) to prove the absence of vertex renormalization in the stochastic NSE.

Vertex renormalization issue - continued

Closely related are other Galilean invariant theories like KPZ and Burgers equation:

- Medina *et. al.*, (Phys. Rev. A**39**, 3053 (1989)) state the vanishing of perturbative corrections to the vertex of the Burgers equation and the KPZ equation is attributed to Galilean invariance.
- Frey and Tauber, (Phys. Rev. E**50**, 1024 (1994)) prove that the vertex in the KPZ equation suffers no renormalization due to Galilean invariance.
- A. L. Barabasi and H. E. Stanley, (*Fractal Concepts in Surface Growth*, Cambridge U. Press, Cambridge, 1995) present nonrenormalization of vertex for KPZ.

Our results show that Galilean invariance ONLY leads to a relationship at zero momentum and frequency for the vertex and so gives NO general information about the vertex. In particular no relation about vertex nonrenormalization follows from Galilean invariance

Vertex renormalization - kinematic argument

(W.D. McComb, Phys. Rev. E**71**, 037301 (2005))

Reynolds decomposition: $V_i = K_i + u_i$

$\langle V_i \rangle = K_i$ - mean velocity ; u_i - fluctuation about mean

Dynamical properties of fluid determine u_i, whereas nonzero K_i emerges when observing fluid outside its comoving frame.

Under Galilean transformation:

$\mathbf{x} = \mathbf{x}' + \mathbf{c}t,\ t = t'\ V_i(\mathbf{x}, t) = V_i'(\mathbf{x}', t') + c_i \Rightarrow$

$K_i = K_i' + c_i$ and $u_i(\mathbf{x}, t) = u_i'(\mathbf{x}', t')$

Equation for K_i vanishes, equation for u_i:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = \nu_0 \nabla^2 u_i$$

Transformation of u_i has no affine component (no constant velocity term c_i), thus no consequences for constraining vertex can arise.

Vertex renormalization - field theory derivation

Effective action:

$$\begin{aligned}\Gamma[\mathbf{V}_{cl},\boldsymbol{\sigma}_{cl},\boldsymbol{\eta}_{cl},\boldsymbol{\eta}^*_{cl}] &= -W[\mathbf{J},\boldsymbol{\Sigma},\boldsymbol{\theta},\boldsymbol{\theta}^*]+\boldsymbol{\theta}^*\cdot\boldsymbol{\eta}_{cl}+\boldsymbol{\theta}\cdot\boldsymbol{\eta}^*_{cl} \\ &+ \int d\mathbf{k}d\omega\,(\mathbf{J}\cdot\mathbf{V}_{cl}+\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma}_{cl}),\end{aligned}$$

where generator of connected diagrams $W = \ln Z_{GF}$

Ward (Slavnov-Taylor) Identities:

$$\begin{aligned}\frac{i}{\xi}\mathbf{V}^{cl}_0\cdot\frac{\delta\Gamma}{\partial\boldsymbol{\eta}_{cl}} &- \frac{i}{\xi}\mathbf{V}^{cl}_0\cdot\frac{\delta\Gamma}{\partial\boldsymbol{\eta}^*_{cl}}+(\eta_{cl}+\eta^*_{cl})_j\int d\mathbf{k}d\omega\,(k_j\frac{\partial V^{cl}_m(\mathbf{k},\omega)}{\partial\omega}\frac{\delta\Gamma}{\delta V^{cl}_m(\mathbf{k},\omega)} \\ &+ k_j\frac{\partial\sigma^{cl}_m(\mathbf{k},\omega)}{\partial\omega}\frac{\delta\Gamma}{\delta\sigma^{cl}_m(\mathbf{k},\omega)}-\delta(\mathbf{k})\delta(\omega)\frac{\delta\Gamma}{\delta V^{cl}_j(\mathbf{k},\omega)}) = 0.\end{aligned}$$

differentiating with respect to $\partial/\partial\eta^{cl}_j\,\delta/\delta V^{cl}_l(\mathbf{k},\omega)\delta/\delta\sigma^{cl}_n(-\mathbf{k},-\omega)$ setting $\boldsymbol{\eta}^{cl}=\boldsymbol{\eta}^*_{cl}=\mathbf{V}^{cl}=\boldsymbol{\sigma}^{cl}=0$, leads to:

$$-k_m\frac{\partial}{\partial\omega}\Gamma^{(1,1)}_{ln}(-\mathbf{k},-\omega;\mathbf{k},\omega)=\Gamma^{(2,1)}_{mln}(\mathbf{0},0;-\mathbf{k},-\Omega;\mathbf{k},\omega).$$

Since Grassman fields only at zero momentum, it implies the above relation only valid at exactly zero momentum transfer.

Extended Galilean invariance (EGI)

(AB, Hochberg, (2008))

Navier-Stokes equation invariant under more general transformation:

$$\begin{aligned}
\mathbf{x}' &= \mathbf{x} - \lambda(t) \\
t' &= t \\
\mathbf{V}'(\mathbf{x}', t') &= \mathbf{V}(\mathbf{x}, t) - \dot{\lambda}(t) \\
\sigma'(\mathbf{x}', t') &= \sigma(\mathbf{x}, t) \\
\Pi'(\mathbf{x}', t') &= \Pi(\mathbf{x}, t) + \mathbf{x}' \cdot \ddot{\lambda}(t)
\end{aligned}$$

$\lambda(t)$ arbitrary time dependent function

Gauge fixed action for EGI:

$$S_{GF} = S[\mathbf{V}, \boldsymbol{\sigma}] + \int dt \left\{ \frac{1}{2\xi} \mathbf{V_0}^2(t) + i\boldsymbol{\eta}^*(t) \frac{d}{dt} \boldsymbol{\eta}(t) \right\}.$$

S contains $\partial \Pi / \partial x_i$ pressure term

Extended Galilean invariance - cont.

Navier-Stokes equation invariant under more general transformation:
BRS symmetry: set $\dot{\boldsymbol{\lambda}}(t) = \mathbf{c}(t) \rightarrow \zeta \dot{\boldsymbol{\eta}}(t)$ - ζ real Grassmann constant

$$\begin{aligned}
\delta_{\mathsf{BRS}} \mathbf{x} &= -\zeta \boldsymbol{\eta}(t) \\
\delta_{\mathsf{BRS}} t &= 0 \\
\delta_{\mathsf{BRS}} \mathbf{V}(\mathbf{x}, t) &= \zeta \eta_k(t) \frac{\partial \mathbf{V}(\mathbf{x}, t)}{\partial x_k} - \zeta \dot{\boldsymbol{\eta}}(t) \\
\delta_{\mathsf{BRS}} \boldsymbol{\sigma}(\mathbf{x}, t) &= \zeta \eta_k(t) \frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial x_k} \\
\delta_{\mathsf{BRS}} \mathbf{V_0}(t) &= -\zeta \dot{\boldsymbol{\eta}}(t) \\
\delta_{\mathsf{BRS}} \boldsymbol{\eta}(t) &= 0 \\
\delta_{\mathsf{BRS}} \boldsymbol{\eta}^*(t) &= \frac{-i}{\xi} \mathbf{V_0}(t) \zeta
\end{aligned}$$

This symmetry leads to a more general Ward identity:

$$\begin{aligned}
&k_n P_{jk}(\mathbf{k}) \left[(\bar{\nu}(\omega + \nu, k) - \bar{\nu}(\nu, k)) k^2 + (\Sigma(\omega + \nu, k) - \Sigma(\nu, k)) \right] \\
&= -\omega \Lambda_{njk}(\mathbf{0}, \omega; \mathbf{k}, \nu; -\mathbf{k}, \omega - \nu)
\end{aligned}$$

In limit $\omega \rightarrow 0$, leads to Ward identity from Galilean invariance Eq. (5)

Conclusion

- The spurious infinity in the standard NSE path integral has been identified and eliminated, to leave a well defined functional.
- Relations between correlation function have been computed using the functional.
- In particular the relation between vertex function and response function, shows, contrary to many claims over the past decades, that no information is contained on the nonrenormalization of the vertex function, except at zero momentum transfer.
- Extended Galilean Invariance has been examined and associated Ward Identities derived.