An E_6 Invariant Action leading to An $SU(5)$ Grand Unification on a Domain Wall

Damien P. George, Archil Kabakhidze, Raymond R. Volkas, Aharon Davidson and Kameshwar C. Wali

Miami 2007: December 13-18, 2007,

- INTRODUCTORY REMARKS
- CLASH OF SYMMETRIES; BASIC IDEA
- $SO(10)$ EXAMPLE TO ILLUSTRATE; WHY WE NEED EXTENSION TO E_6
- E_6 INVARIANT ACTION
- SUMMARY AND CONCLUSIONS

INTRODUCTORY REMARKS

- DEFAULT MECHANISM: CONVENTIONAL THEORY OF SPONTANEOUS SYMMETRY BREAKING (SM AND ITS EXTENSIONS)

- OTHER TYPES OF SOLUTIONS TO HIGGS FIELDS THAT CAN SERVE AS STABLE, STATIC BACKGROUND FIELDS: TOPOLOGICAL SOLITONS, SUCH AS KINKS, STRINGS AND MONOPOLES. HOWEVER, THEIR SPATIAL NON-HOMOGENEITY FORBIDS THEM TO BE USED AS BACKGROUND FIELDS IN OUR 3+1 DIMENSIONAL UNIVERSE FOR WHICH THERE SEEMS TO BE STRONG EVIDENCE FOR ITS LARGE SCALE HOMOGENEITY

  THIS OBJECTION,HOWEVER, DOES NOT APPLY TO BRANE-WORLD MODELS, SINCE THE NON-TRIVIAL SPACIAL DEPENDENCE OF THE HIGGS FIELDS CAN BE RESTRICTED TO EXTRA DIMENSION COORDINATES ONLY

- WE HAVE EXPLORED THIS IDEA IN RECENT YEARS, PRESENTING NON-TRIVIAL, BUT STILL IN THE NATURE OF TOY MODELS:

- A MODEL WITH THREE HIGGS TRIPLETS

- DOMAIN WALL SOLUTIONS WITH ABELIAN GAUGE FIELDS

- A GLOBAL $U(1)\otimes U(1)$ MODEL WITH RANDALL-SUNDRUM-LIKE GRAVITY. A MODEL WITH TWO COMPLEX SCALAR FIELDS IN A FIVE DIMENSIONAL SPACE-TIME.

- THE PRESENT WORK AIMS AT A REALISTIC MODEL ALTHOUGH MUCH WORK IS NEEDED TO ACHIEVE THAT END.

- CLASH OF SYMMETRIES; BASIC IDEA

- CONSIDER, FOR EXAMPLE, AN EXTRA DIMENSION, COORDINATE w AND TOPOLOGICALLY STABLE HIGGS FIELD CONFIGURATIONS $\Phi_i(w)$, SOME OF WHICH HAVE KINK FORM WITH RESPECT TO w. PATTERN OF SYMMETRY BREAKING BECOMES A FUNCTION OF w.

- BRAIN, OUR $3+1$ DIMENSIONAL UNIVERSE LOCATED AT $w=0$ WITH PHYSICAL FIELDS CONFINED TO THE BRANE. UNBROKEN SYMMETRY IS THE STABILITY GROUP OF $\Phi_i(w=0)$, SAY A SUBGROUP $H(w=0)$ OF SOME INTERNAL SYMMETRY GROUP G

- THE STABILITY GROUP OF $\Phi_i(0<|w|<\epsilon), H(w \sim \epsilon)$ IS DIFFERENT FROM $H(w=0)$, THEN A RICH EFFECTIVE SYMMETRY BREAKING IS POSSIBLE ON THE BRANE

- THIS HAPPENS WHEN ISOMORPHIC GROUPS $H(|w|=\infty)$ LEFT UNBROKEN AT $|w|=\infty$ CAN BE DIFFERENTLY EMBEDDED IN THE PARENT GROUP G

  THE BREAK DOWN AT FINITE w IS *THE INTERSECTION OF THE ASYMPTOTIC STABILITY GROUPS,*

$$H(|w|<\infty) = H(-\infty) \cap H(+\infty) \equiv H_{clash}$$

- THUS IN GENERAL, IF THE VACUUM MANIFOLD IS DISCONNECTED AND CONTAINS DISTINCT VACUUM STATES, THE FIELD CAN SETTLE INTO DIFFERENT MINIMIZING VACUUM STATES IN DIFFERENT SPATIAL DIMENSIONS.

- STABLE SOLITONIC OR KINK-LIKE DOMAIN WALL CONFIGURATIONS CAN EXIST IF THE VACUUM MANIFOLD HAS THE APPROPRIATE TOPOLOGY.

- IF OUR (3+1) DIMENSIONAL UNIVERSE IS A BRANE OR A DOMAIN WALL, SUCH A SITUATION CAN ARISE, NAMELY, A GLOBAL

  $G_{cts} \otimes G_{discrete}$ IS SPONTANEOUSLY BROKEN TO $H_{cts} \otimes H_{discrete}$,

  WHERE H_{cts} CAN BE EMBEDDED IN SEVERAL DIFFERENT WAYS IN THE PARENT G_{cts} AND $H_{discrete} \subset G_{discrete}$

- AS A CONSEQUENCE, A CERTAIN CLASS OF DOMAIN WALL SOLUTIONS CONNECTS TWO VACUA THAT ARE INVARIANT UNDER DIFFERENTLY EMBEDDED H_{cts} SUBGROUPS

AND

- THERE IS AN ENHANCED SYMMETRY BREAKDOWN TO THE INTERSECTION OF THE TWO SUBGROUPS ON THE BRANE OR DOMAIN WALL

- IN THE BRANE LIMIT, H_{cts} PREVAILS IN THE BULK, BUT SMALLER INTERSECTION SYMMETRY ON THE BRANE ITSELF.

  WE CALL THIS PHENOMENON

"CLASH" OF SYMMETRIES!!

Confluence of this Clash-of-Symmetry (CoS)combined with matter fields confinement ideas (Dvali-Shifman) provides a good basis for realistic models

- EXAMPLE OF $SO(10)$ MODEL

- Consider a most general quartic Higgs potential with χ, a scalar multiplet belonging to the adjoint representation (45) of $SO(10)$

$$\mathcal{V} = 1/2\mu^2 Tr(\chi^2) + 1/4\lambda_1 Tr(\chi^2)^2 + 1/4\lambda_2 Tr(\chi^4)$$

$$\chi = f_\alpha \hat{X}^\alpha$$

where the $\hat{X}$'s are matrix representations of the generators in the fundamental of $SO(10)$ while the f_α's are the components of the adjoint multiplet.

- An accidental Z_2 symmetry because of the absence of cubic term in the potential

  Global minimization of such a potential is well-known. Using SO(10)transformation, we can write

$$\chi = \text{diag}(f_1\,\epsilon\,,\ f_2\,\epsilon\,,\ f_3\,\epsilon\,,\ f_4\,\epsilon\,,\ f_5\,\epsilon), \qquad (1)$$

where the f_i are real and

$$\epsilon \equiv i\,\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (2)$$

The f_i are real and the five independent ones correspond to the five independent generators in the Cartan subalgebra of $SO(10)$

In this basis,

$$V = -\mu^2 \sum_{i=1}^{5} f_i^2 + \lambda_1 \left(\sum_{i=1}^{5} f_i^2 \right)^2 + \frac{1}{2} \lambda_2 \sum_{i=1}^{5} f_i^4. \qquad (3)$$

For $\lambda_2 > 0$, the global minima of V are at

$$f_i^2 = \frac{\mu^2}{10\lambda_1 + \lambda_2} \quad \equiv f_{\text{min}}^2 \ \forall i, \qquad (4)$$

The unbroken subgroup is $H = U(5)$.

Define $f_{\text{min}} \equiv \sqrt{\mu^2/(10\lambda_1 + \lambda_2)}$

The values of f_i at the minima are specified up to a sign that can be chosen independently for each component

$$f_i = \pm f_{\text{min}}. \qquad (5)$$

Different choices for these signs correspond to two features: different embeddings of $U(5)$ in $SO(10)$ and also a choice of which Z_2 sector the minimum lies in.

For domain wall configurations, suppose at $y = -\infty$, we choose as our boundary condition

$$\chi(-\infty) = -f_{\text{min}}^{(5)} \equiv -f_{\text{min}}\,\text{diag}(\epsilon\,,\,\epsilon\,,\,\epsilon\,,\,\epsilon\,,\,\epsilon). \tag{6}$$

Then at $y = +\infty$, we have the following choices:

$$\chi(+\infty) = \begin{cases} f_{\text{min}}^{(5)} & \equiv & f_{\text{min}}\,\text{diag}(\epsilon\,,\,\epsilon\,,\,\epsilon\,,\,\epsilon\,,\,\epsilon) \\ f_{\text{min}}^{(3,2)} & \equiv & f_{\text{min}}\,\text{diag}(\epsilon\,,\,\epsilon\,,\,\epsilon\,,\,-\epsilon\,,\,-\epsilon) \\ f_{\text{min}}^{(4,1)} & \equiv & f_{\text{min}}\,\text{diag}(\epsilon\,,\,-\epsilon\,,\,-\epsilon\,,\,-\epsilon\,,\,-\epsilon) \end{cases} . \tag{7}$$

The three vacua in Eq. (7) are invariant under differently-embedded subgroups of $SO(10)$: $U(5)_1$, $U(5)_2$ and $U(5)_3$.

The superscripts (5), (3,2) and (4,1) denote the numbers of plus and minus signs in the VEVs. But they also describe the unbroken symmetry of the domain wall at finite y, respectively

$$U(5), \qquad U(3)\otimes U(2) \quad \text{and} \quad U(4)\otimes U(1), \tag{8}$$

The ansatz for domain wall configurations that interpolate between the stated boundary conditions:

$$\chi(y) = h(y)\chi(-\infty) + g(y)\chi(+\infty),$$

where the functions h and g obey self-evident boundary conditions.

The first configuration interpolates between $-f_{\text{min}}$ and $+f_{\text{min}}$ for all components $f_i(y)$ It breaks $SO(10)$ to $U(5)_1$ at all values of y, because the relative magnitudes of the components are always the same at a given y. It is a non-CoS domain wall.

The second configuration has an equal-magnitude 3×3 block (of 2×2 submatrices), and an equal-magnitude 2×2 block. The unbroken symmetry is then

$$U(3) \otimes U(2) = U(5)_1 \cap U(5)_2. \tag{9}$$

Similarly, the third configuration's block structure leads to the unbroken symmetry

$$U(4) \otimes U(1) = U(5)_1 \cap U(5)_3. \tag{10}$$

The Euler-Lagrange equations

$$f_i'' = 2\left[-\mu^2 + 2\lambda_1 \sum_{j=1}^{5} f_j^2\right] f_i + 2\lambda_2 f_i^3, \qquad (11)$$

with the three types of boundary conditions above can be solved numerically.

That solutions exist follows from the fact that one finds analytic solution for solution $\lambda_1 = 0$ slice through parameter space.

For the first boundary condition choose

$$f_i(y) = f_{\text{min}} \tanh(\mu y) \ \forall i \qquad (12)$$

For the second boundary condition choose

$$f_i(y) = f_{\text{min}} \tanh(\mu y) \text{ for } i = 1,2,3 \text{ and } f_i(y) = f_{\text{min}} \text{ for } i = 4,5 \qquad (13)$$

For the third boundary condition choose

$$f_i(y) = f_{\text{min}} \tanh(\mu y) \text{ for } i = 1 \text{ and } f_i(y) = f_{\text{min}} \text{ for } i = 2,3,4,5 \qquad (14)$$

The surface energy densities are in the ratios 5 : 3 : 1 from the first to the third, implying the last gives the topologically stable configuration.

From a Dvali-Shifman point of view, this stable configuration has an unbroken $SU(4)$ on the brane that is embedded in $SU(5)_1$ on the $y < 0$ side of the wall, and $SU(5)_3$ on the $y > 0$ side.

Thus if the Dvali-Shifman mechanism is correct,the $SU(4)$ gauge bosons are localized to the wall since by assumption both, $SU(5)_1$ and $SU(5)_3$ are in confinement phase in their respective bulk regions.

This establishes the connection between clash-of-symmetries and Dvali-Shifman by way of an explicit rigorously worked-out solution. However, it is only a toy model since it is not what we want phenomenologically.

Comments about the other alternative,

$$U(3) \otimes U(2) = SU(3) \otimes SU(2) \otimes U(1) \otimes U(1) \qquad (15)$$

on the brane is closer to a realistic model. It has the right ingredients.

The analytical solution says that it is dynamically unstable. It could be, however, stable in a different region of parameter space. But closer examination shows that neither the photon nor the Z^0 are fully confined.

The structure corresponds to localised gluons and $W^{\pm}$ bosons, but semi-delocalised photons and Z^0's.

It is close enough to reality.But for a more realistic model, we need to extend the symmetry on the brane.

Requirements:

- One needs a full $SU(5)$, with the physical hypercharge identified with one of its generators.

- This brane-$SU(5)$ must be a subgroup of confining non-Abelian groups on both sides of the domain wall.

- These two features automatically arise when we upgrade from $SO(10)$ to E_6 as the symmetry of the action and choose a suitable Higgs potential that will induce

$$E_6 \to SO(10) \otimes U(1), \tag{16}$$

which is a maximal subgroup.

- Furthermore, we find a pair of embeddings, $SO(10) \otimes U(1)_E$ and $SO(10)' \otimes U(1)_{E'}$, that provides a domain wall solution that interpolates between VEVs that break E_6 to these different but isomorphic subgroups on opposite sides of the wall.

  The symmetry on the wall is then

$$[\,SO(10) \otimes U(1)_E\,] \cap [\,SO(10)' \otimes U(1)_{E'}\,] = SU(5) \otimes U(1)_E \otimes U(1) \tag{17}$$

We are looking at the symmetry breaking pattern,

$$E_6 \to SO(10) \otimes U(1)_E \to [\, SU(5) \otimes U(1)_X \,] \otimes U(1)_E, \quad (18)$$

Two representations of interest, the fundamental 27 and the adjoint 78:

The fundamental 27-dimensional representation of E_6 branches as per

$$\begin{aligned} 27 &\to 1(4) + 10(-2) + 16(1) \\ &\to 1^{(0,4)} + [\, 5^{(2,-2)} + 5^{*(-2,-2)} \,] + [\, 1^{(-5,1)} + 5^{*(3,1)} + 10^{(-1,1)} \quad (19) \end{aligned}$$

The second embedding is revealed by considering the linear combinations

$$X' = -\frac{1}{4}(X + \sqrt{15}E), \qquad E' = \frac{1}{4}(-\sqrt{15}X + E) \quad (20)$$

that correspond to

$$E_6 \to SO(10)' \otimes U(1)_{E'} \to [\, SU(5) \otimes U(1)_{X'} \,] \otimes U(1)_{E'}. \quad (21)$$

The adjoint rep. 78 branches into

$$\begin{aligned} 78 &\to 1(0) + 45(0) + 16(-3) + 16^*(3) \quad (22) \\ &\to 1^{(0,0)}_{(0,0)} \\ &+ [\, 1^{(0,0)}_{(0,0)} + 10^{(4,0)}_{(-1,-3)} + 10^{*(-4,0)}_{(1,3)} + 24^{(0,0)}_{(0,0)} \,] \\ &+ [\, 1^{(-5,-3)}_{(5,3)} + 5^{*(3,-3)}_{(3,-3)} + 10^{(-1,-3)}_{(4,0)} \,] \\ &+ [\, 1^{(5,3)}_{(-5,-3)} + 5^{(-3,3)}_{(-3,3)} + 10^{*(1,3)}_{(-4,0)} \,] \quad (23) \end{aligned}$$

The $SU(5)$ adjoint $24^{(0,0)}_{(0,0)}$ is common to both $SO(10)$ embeddings, a necessary condition for being in the intersection of the two.

The two $1^{(0,0)}_{(0,0)}$ multiplets play important roles.

- Giving a VEV to the 1(0) breaks E_6 to $SO(10) \otimes U(1)_E$.

- A VEV for the second singlet breaks E_6 to $SO(10)' \otimes U(1)_{E'}$

- A clash-of-symmetries kink interpolates between these two VEVs imposed as boundary conditions. At $|y| < \infty$, both $SU(5) \otimes U(1)^2$ singlet components of the 78 have nonzero values, and this is precisely why the configuration breaks E_6 to the intersection of the two subgroups.

To realize the above configuration, we choose the Higgs potential using the scalar multiplet χ represented by

$$\chi = f_\alpha \widehat{X}^\alpha, \quad \alpha = 1, \ldots, 78 \tag{24}$$

where $\widehat{X}$'s are matrix representations of the generators for the 27 of E_6, and the f's are the field components.

The two of the seventy-eight fields we are concerned with are those associated with (E, E'), equivalently (X, E) or (X', E') depending on what basis we choose for the Lie algebra,namely,

$$\chi = f_E E + f_X X \equiv \tilde{f}_E E + f_{E'} E' \tag{25}$$

with

$$\tilde{f}_E \equiv f_E + \frac{f_X}{\sqrt{15}}, \qquad f_{E'} \equiv -\frac{4 f_X}{\sqrt{15}}, \tag{26}$$

The VEVs we want for the boundary conditions are

$$(\tilde{f}_E, f_{E'}) = v(1, 0), \tag{27}$$

which corresponds to $E_6 \to SO(10) \otimes U(1)_E$. The other VEV is

$$(\tilde{f}_E, f_{E'}) = -v(0, 1) \tag{28}$$

which gives $E_6 \to SO(10)' \otimes U(1)_{E'}$.

- Comment about the relative minus sign:
- Breaking of the discrete symmetry to ensure the domain walls are topologically non-trivial
- It has a remarkable consequence for zero-mode localization

  In terms of the (X, E) basis, these same VEVs are

$$(f_X, f_E) = v(0,1) \quad \text{and} \quad v\left(\frac{\sqrt{15}}{4}, -\frac{1}{4}\right), \qquad (29)$$

respectively.

We postulate a Higgs potential with these two VEVs as degenerate global minima. The Higgs potential is constructed out of adjoint invariants, which are simply the nth order Casimir invariants

$$I_2, \; I_5, \; I_6, \; I_8, \; I_9, \; I_{12}, \tag{30}$$

The fact that $I_{5,9}$ are nonzero means that the discrete Z_2 of Eq.(**??**) is *not a subgroup* of E_6, because imposing it eliminates the otherwise present odd-power invariants.

It is sensible to truncate the Higgs potential at order-six:

$$V = -\lambda_1 I_2 + \lambda_2 (I_2)^2 - 2304\kappa I_6 + \frac{4}{3}\lambda_3 (I_2)^3 \tag{31}$$

where some peculiar numbers and signs have been inserted for later numerical convenience.

The potential V is a complicated sextic in seventy-eight fields. But for a good qualitative understanding,it is sufficient to restrict to the two Cartan-subalgebra generators, f_E and f_X

The traces of these generators are given by

$$I_2 = \frac{1}{2}(f_E^2 + f_X^2), \tag{32}$$

$$I_6 = \frac{1}{2304}\left(f_E^6 + 5f_E^4 f_X^2 + 7f_E^2 f_X^4 - \frac{48}{5\sqrt{15}} f_E f_X^5 + \frac{83}{25} f_X^6\right) \tag{33}$$

By the standard procedure we find that the value of V at the minima is given by

$$V_{\text{min}} = -\frac{1}{3}\sqrt{\frac{\lambda_1^3}{\lambda_3 - 22\kappa}}. \tag{34}$$

The latter must be subtracted from the potential

$$V \to V - V_{\text{min}} \tag{35}$$

to produce finite energy-densities for the domain wall configurations.

Having found the global minima, we solve the Euler-Lagrange equations and find numerical solutions for CoS domain walls interpolating between $(10, +)$ at $y = -\infty$ and $(10', -)$ at $y = +\infty$ with two different parameter choices are displayed

(Contour Plot of the Higgs Potential and Domain wall solutions)

- Fermion zero-mode localization

- The CoS E_6 domain wall solutions are a good starting point for $3+1$-d theories for localised fields.

- With Ψ in the 27 of E_6, coupled to the adjoint scalar χ, we solve for five-dimensional Dirac equation with the CoS solution for χ

$$i\Gamma^M \partial_M \Psi^{(X,E)}(x^\mu, y) - h[f_X(y)X + f_E(y)E]\Psi^{(X,E)}(x^\mu, y) = 0. \quad (36)$$

- We find that the localized spectrum consists of Left-handed zero-modes in the $SU(5)$ representation

$$5^* \oplus 10 \oplus 1 \oplus 1, \quad (37)$$

- The spectrum represents a single standard family with two singlet neutrinos.

- Apart from the extra singlets, all exotic fermions in the 27 of E_6 are delocalized.

- The spectrum resembles a usual $SO(10)$ family with an additional singlet. However, the 5^* and the 10 do not come from a 16 of either $SO(10)$ or $SO(10)'$

- SUMMARY AND CONCLUSIONS

- Establishes a general connection between the Clash-of-Symmetries (CoS) mechanism for simultaneous brane-creation and internal symmetry breaking

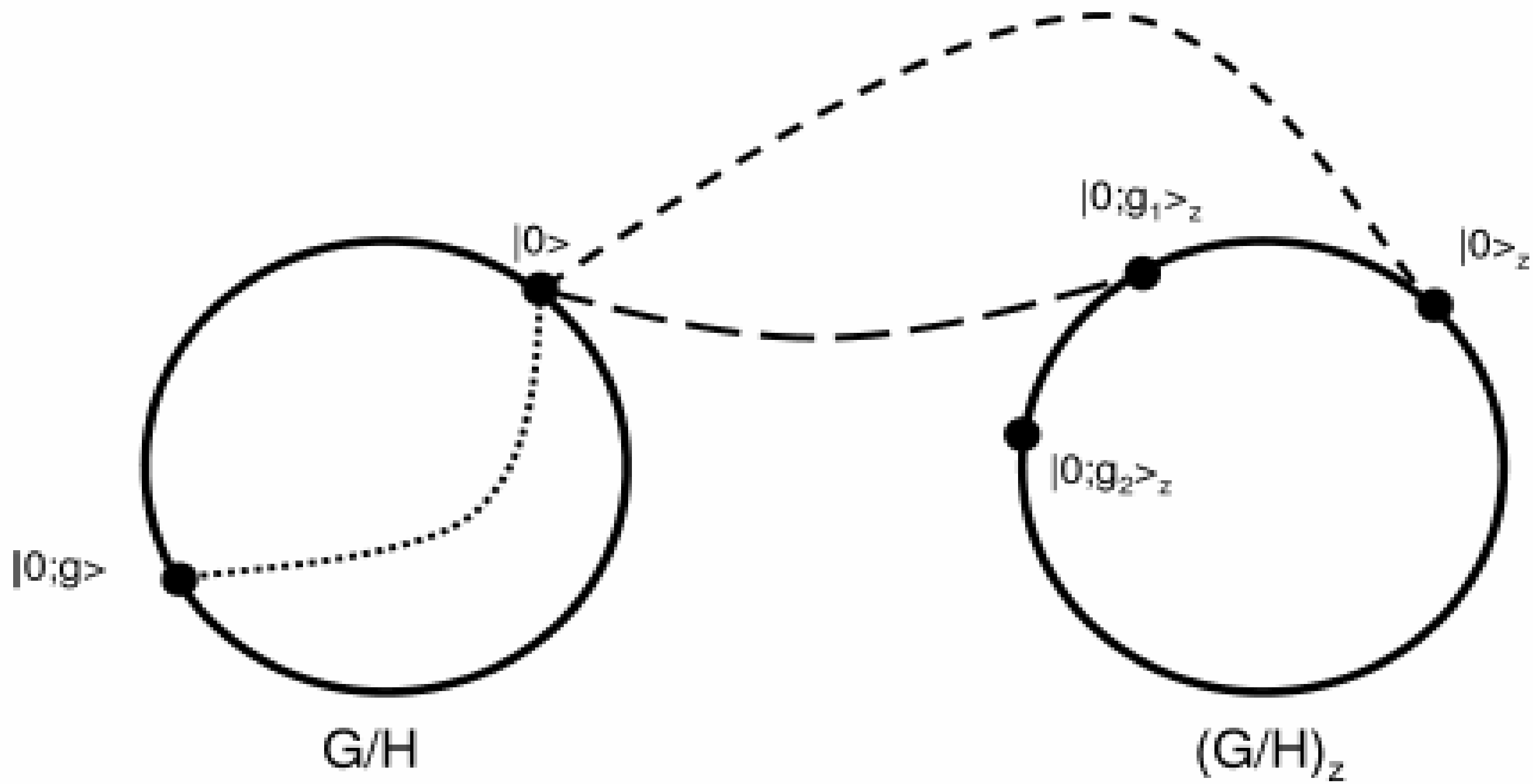
- In the E_6 context, CoS idea leads to interesting results concerning localization of gauge and fermionic fields. Confluence of CoS and Dvali-Shifman mechanism.

- More specifically, we have found a domain wall solution in an E_6 adjoint-Higgs model that produces an $SU(5) \otimes U(1)^2$ symmetry on the wall itself.

- In one half of the bulk, the symmetry is enhanced to $SO(10) \otimes U(1)$, while in the other half of the bulk the enhancement is to $SO(10)' \otimes U(1)'$, both the groups being differently-embedded but isomorphic subgroups of E_6.

- Because the brane-$SU(5)$ is contained in both $SO(10)$ and $SO(10)'$, the Dvali-Shifman localisation of its gauge bosons follows.

- The simplest possible mechanism for fermion zero-mode localisation produces a realistic spectrum, an outcome that depends on the generic features of our domain wall configuration.


|0>
|0;g>
|0;g1>z
|0>z
|0;g2>z
G/H
(G/H)z

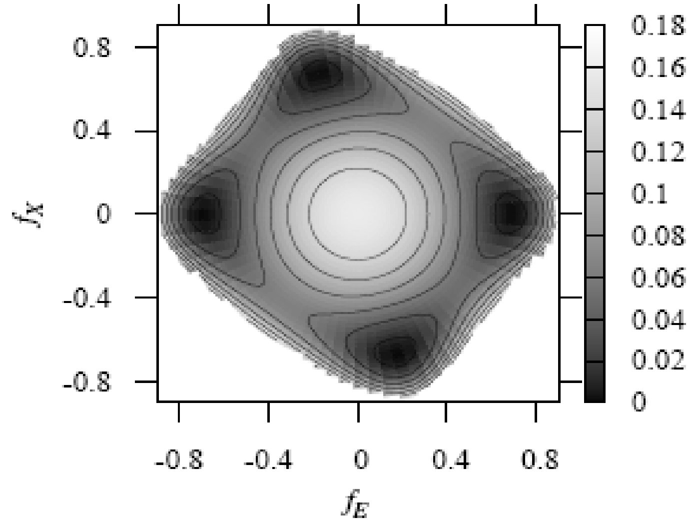


FIG. 3: Contour plot of the Higgs potential as a function of the two field components f_E and f_X. The parameters used are $\kappa = 0.8$, $\lambda_1 = 1.0$, $\lambda_2 = 0$, $\lambda_3 = 22.0$. The darkest regions are the global minima in the order $(10, +)$, $(10', -)$, $(10, -)$ and $(10', +)$ reading anticlockwise from the rightmost minimum. The light area near the origin is a local maximum.

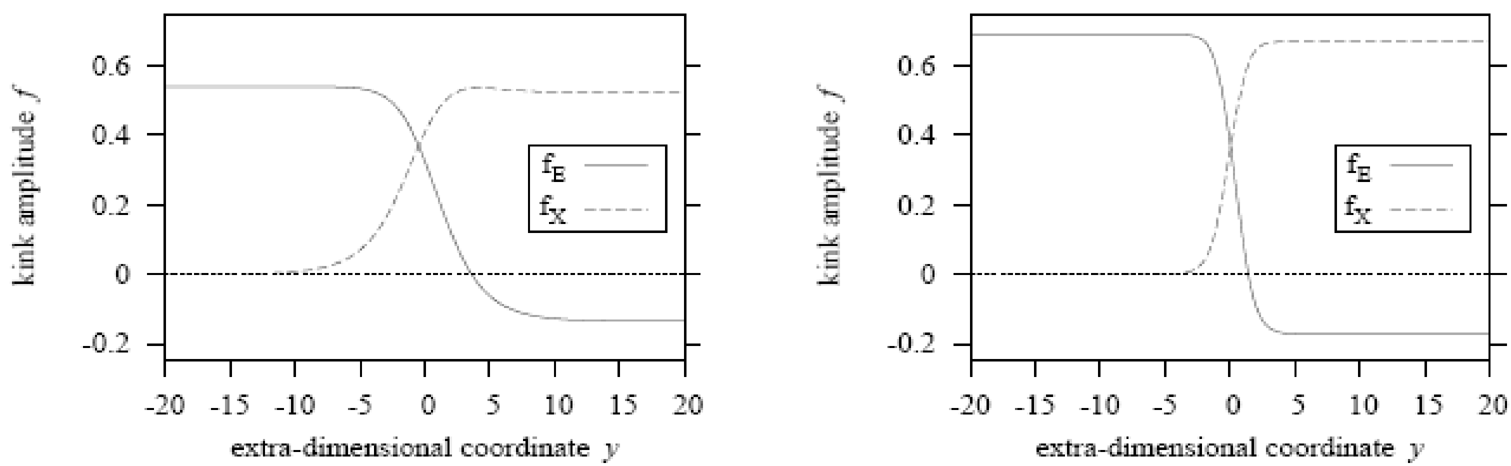


FIG. 4: Clash-of-symmetries domain wall solutions interpolating between $(10, +)$ at $y = -\infty$ and $(10', -)$ at $y = +\infty$. The parameters used in the left plot are $\kappa = 0.2$, $\lambda_1 = 1.5$, $\lambda_2 = 0$, $\lambda_3 = 22.0$; those in the right plot are $\kappa = 0.8$, $\lambda_1 = 1.0$, $\lambda_2 = 0$, $\lambda_3 = 22.0$.

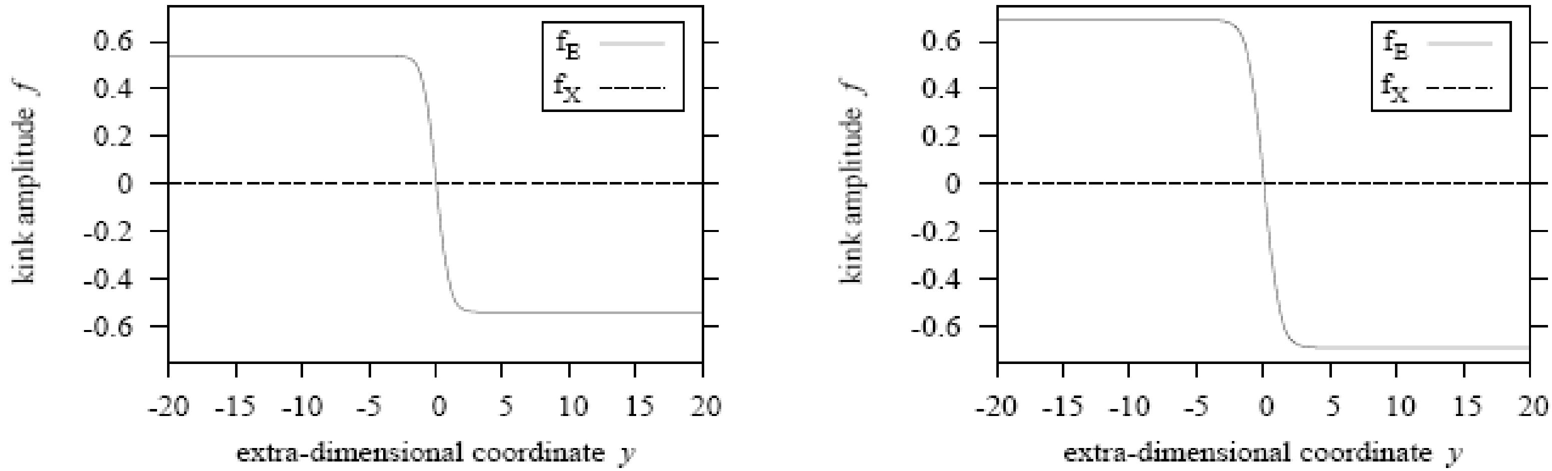


FIG. 5: Non-clash-of-symmetries domain wall solutions interpolating between $(10, +)$ at $y = -\infty$ and $(10, -)$ at $y = +\infty$. The parameters used in the left plot are $\kappa = 0.2$, $\lambda_1 = 1.5$, $\lambda_2 = 0$, $\lambda_3 = 22.0$; those in the right plot are $\kappa = 0.8$, $\lambda_1 = 1.0$, $\lambda_2 = 0$, $\lambda_3 = 22.0$.

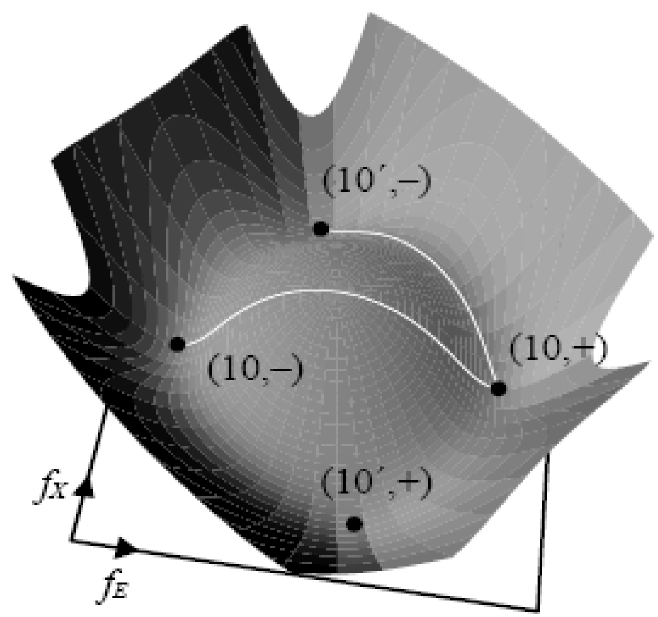


FIG. 6: Three-dimensional plot of the Higgs potential as a function of the two field components f_E and f_X. The white lines show the clash-of-symmetries domain wall (topmost) and the non-CoS domain wall (bottommost). The parameters used are $\kappa = 0.8$, $\lambda_1 = 1.0$, $\lambda_2 = 0$, $\lambda_3 = 22.0$.

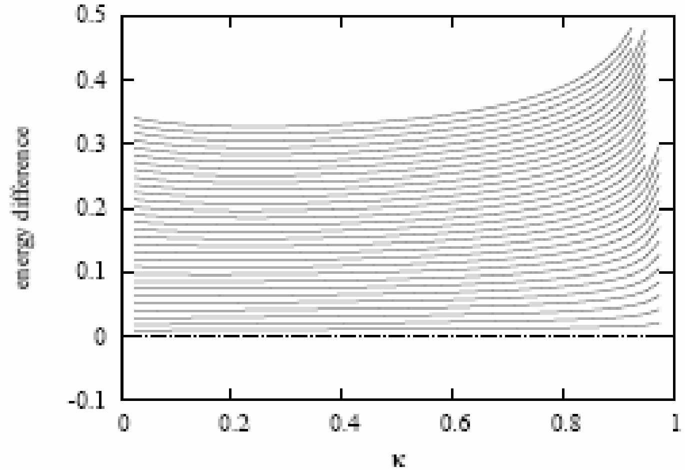


FIG. 7: The difference in energy densities between the non-CoS and CoS domain wall solutions, $E_{\mathrm{non-CoS}} - E_{\mathrm{CoS}}$. We have numerically scanned through the parameter space with $0 < \kappa < 1$ along the horizontal axis, and each successive curve represents a different λ_1, beginning at $\lambda_1 = 0.05$ at the bottom and increasing in steps of 0.05 to $\lambda_1 = 1.5$ at the top. The energy difference is always positive, so the CoS domain wall has a lower energy. We set $\lambda_2 = 0$ for simplicity.

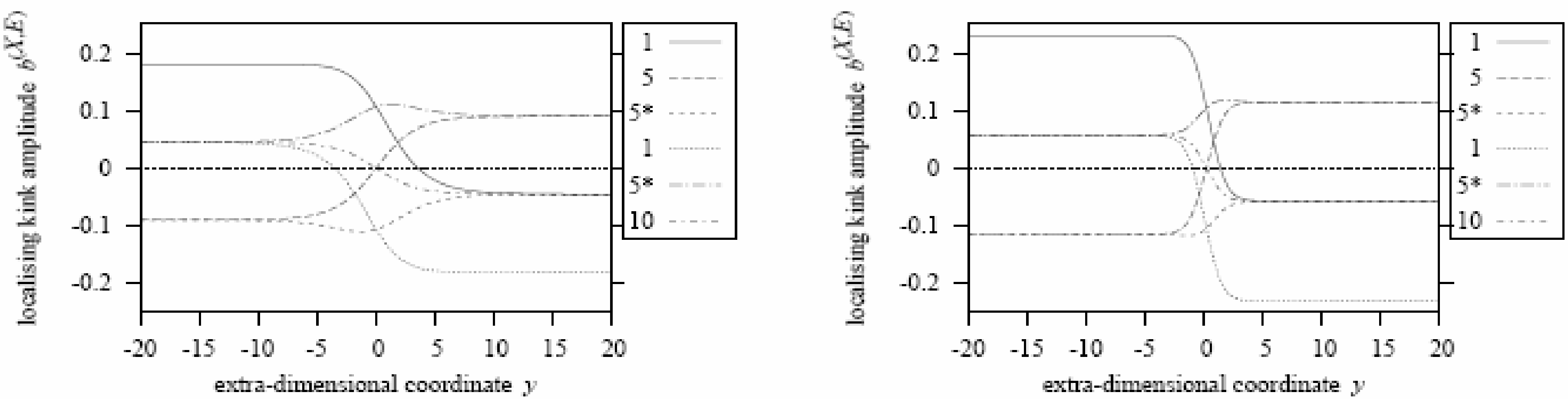


FIG. 8: Clash-of-symmetries fermion localising profiles interpolating between $(10, +)$ at $y = -\infty$ and $(10', -)$ at $y = +\infty$. The parameters used in the left plot are $\kappa = 0.2$, $\lambda_1 = 1.5$, $\lambda_2 = 0$, $\lambda_3 = 22.0$; those in the right plot are $\kappa = 0.8$, $\lambda_1 = 1.0$, $\lambda_2 = 0$, $\lambda_3 = 22.0$. The top to bottom order of the $SU(5)$ fermion multiplets in the box on the right matches the order in Eq. (4.48).

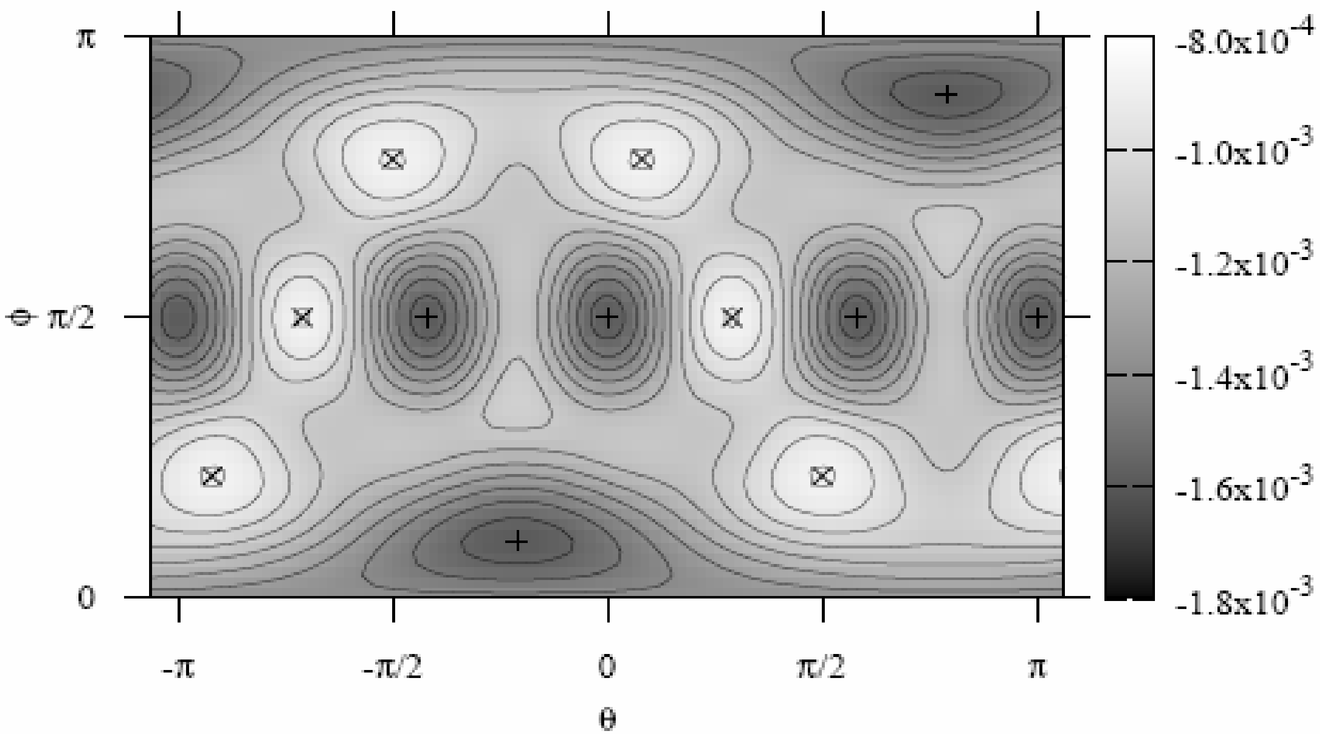


FIG. 9: Contour plot of $-I_6/r^6$ as a function of θ and ϕ. The degenerate global minima breaking E_6 to various $SO(10) \otimes U(1)$ subgroups are marked with + signs, while the local maxima are indictaed with $\times$ signs. The row of minima along $\phi = \pi/2$ correspond to the global minima displayed in Fig. 2.