

Multiloop Gluon Amplitudes

and

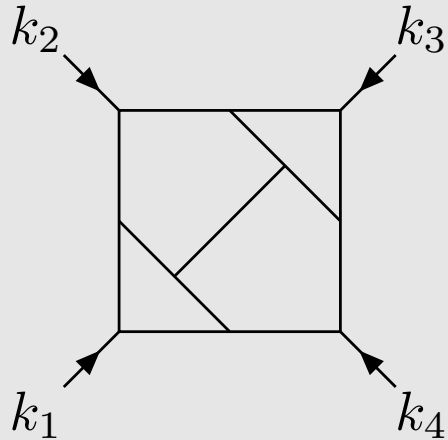
AdS/CFT

Anastasia Volovich

Brown University

with F. Cachazo and M. Spradlin

Preview of the Results



The meat of this work is a very efficient new algorithm for extracting certain quantities (the **cusp** and the **collinear anomalous dimensions**) from L -loop gluon amplitudes in $\mathcal{N} = 4$ Yang Mills.

The method does have wider applicability, but our interest in these particular calculations stems from their important impact on studies of **integrability** in $\mathcal{N} = 4$ Yang Mills. Concrete calculations of anomalous dimensions are needed to shed light on certain structures and in particular to test various all-loop conjectures in the literature.

We computed **four-loop anomalous dimensions** to a very high precision.

Motivations

- **Gluon scattering amplitudes** in QCD and supersymmetric gauge theories are very difficult to compute, so this is a fertile ground for new insights and methods.
- Over the past several years we have learned a lot about remarkably rich mathematical structures in Yang-Mills theories.
- On one hand, in the context of AdS/CFT a lot of work has been done exploring Yang-Mills **integrable structures** and computing anomalous dimensions. Exact 'S-matrix' for planar $\mathcal{N} = 4$ Yang-Mills theory.
- On the other hand, following the discovery of **twistor string theory**, we have seen a lot of progress in Yang-Mills scattering amplitudes. Of particular interest are iterative relations for planar $\mathcal{N} = 4$ Yang-Mills amplitudes.
- **Cusp anomalous dimension is a player in both games.**

Introduction: Yang-Mills Theory

Of course there are many reasons to be interested in YM theory.

- The unique theory of interacting vector bosons
- A great deal of interesting mathematics
- Not to mention, of course, QCD and the 'Real World'!

The journey towards an analytic solution of this important and rich theory has been long and profitable.

Introduction: Yang-Mills Theory

Of course there are many reasons to be interested in YM theory.

- The unique theory of interacting vector bosons
- A great deal of interesting mathematics
- And, of course, QCD and the 'Real World'!

The journey towards an analytic solution of this important and rich theory has been long and profitable.

Like in many areas of physics, if we can't solve the theory we're most interested in, we look for a simpler, similar model that we can solve!

This leads us to consider the $\mathcal{N} = 4$ supersymmetric version of the theory, which has even richer mathematical structure and of course is of great independent interest since it is a theory of quantum gravity.

Yang-Mills and Integrability

Integrability is a very powerful computational tool.

Anomalous dimensions of operators $\Delta(\lambda)$ should be determinable by **integrability**—a technology familiar from the study of spin chains.

Single-trace operator corresponds to a spin chain configuration [Lipatov] [Faddeev, Korchemsky] [Minahan & Zarembo].

Anomalous dimension computation reduces to diagonalization of the spin chain Hamiltonian.

An important property of integrable theories is that they can be solved (i.e., the eigenvalues of the Hamiltonian can be found) once we know **the two-particle S-matrix**.

The Guess S-Matrix of SYM

Beisert showed that the matrix structure of the **two-particle S-matrix** is completely fixed by supersymmetry.

The only remaining freedom is an overall phase factor

$$S_0(\lambda; p_1, p_2)$$

which depends on the momenta of the two magnons but not on their species.

A conjecture for this phase factor, which satisfies many desirable properties, has been proposed. [**Beisert, Eden, Staudacher; Beisert, Hernandez, Lopez**].

This guess S-matrix via Bethe ansatz equations gives the entire spectrum of $\mathcal{N} = 4$ Yang-Mills in large J limit.

The Cusp Anomalous Dimension

The cusp anomalous dimension

$$f(\lambda) = \sum_{L=1}^{\infty} f^{(L)}(\lambda) = 4\lambda - 4\zeta(2)\lambda^2 + (4\zeta(2)^2 + 12\zeta(4))\lambda^3 + \mathcal{O}(\lambda^4)$$

governs the behavior of twist-two operators in the limit of very large spin:

$$\Delta (\text{Tr}[ZD^S Z]) = S + f(\lambda) \log S + \mathcal{O}(S^0), \quad S \gg 1.$$

This quantity has long played an important role in quantitative checks of AdS/CFT [Gubser, Klebanov, Polyakov] [Frolov, Tseytlin].

(Not to mention QCD, where the analogous anomalous dimension quantifies the leading violation to Bjorken scaling.)

The BES/BHL S-matrix implies a **(complicated!) integral equation** for $f(\lambda)$ valid for all λ .

The Cusp Anomalous Dimension

At strong coupling, the equation is difficult to handle analytically, but numerical results show that the solution exhibits remarkable agreement with AdS/CFT

$$f(\lambda) = 4\sqrt{\frac{\lambda}{16\pi^2}} - \frac{3 \log 2}{\pi} + \mathcal{O}(1/\sqrt{\lambda})$$

[Gubser, Klebanov, Polyakov; Frolov, Tseytlin] [Alday, Arutyunov, Benna, Eden, Klebanov; Benna, Benvenuti, Eden, Klebanov, Scardicchio; Basso, Korchemsky, Kotanski; Roiban, Tseytlin]

At weak coupling, the perturbative solution leads to the four-loop prediction

$$f(\lambda) = 8 \left(\frac{\lambda}{16\pi^2} \right) - \frac{8\pi^2}{3} \left(\frac{\lambda}{16\pi^2} \right)^2 + \frac{88\pi^2}{45} \left(\frac{\lambda}{16\pi^2} \right)^3 - \left(\frac{584\pi^2}{315} + 128\zeta_3^2 \right) \left(\frac{\lambda}{16\pi^2} \right)^4 + \mathcal{O}(\lambda^5)$$

The rest of the talk will be devoted to the calculation of this and other quantities from **perturbative scattering amplitudes in $\mathcal{N} = 4$ YM.**

Gluon Scattering Amplitudes

We have learned that Feynman diagrams are not the most efficient way to calculate scattering amplitudes: too messy+too many terms+hide the structure of amplitudes.

There has been a lot of progress on tree amplitude calculations stimulated by **twistor string theory**. [Witten]

MHV rules, recursion relations, etc. [Cachazo, Svrcek, Witten] [Britto, Cachazo, Feng, Witten] [Roiban, Spradlin, AV] [Brandhuber, Spence, Travaglini] [Dixon, Glover, Khoze] [Bern, Dixon, Kosower] [Badger, Glover, Khoze] [Berkovitz, Motl] [Gukov, Motl, Neitzke] [many others]

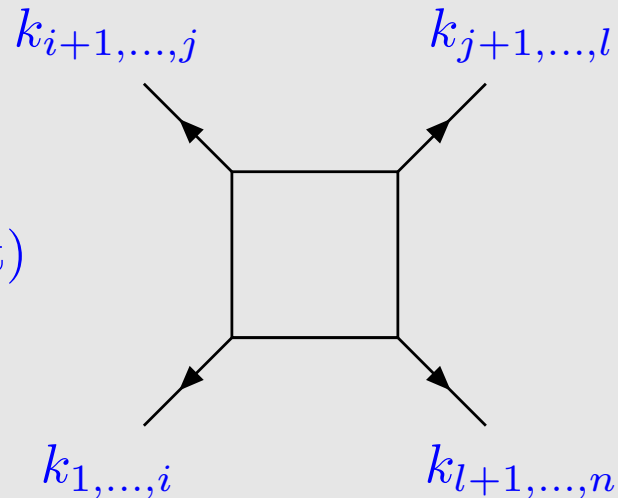
All tree level perturbative amplitudes are under control.

One-Loop Amplitudes

In the $\mathcal{N} = 4$ theory, all one-loop integrals which appear in any Feynman diagram calculation can be reduced to a **set of scalar box integrals**.

In other words, scalar box integrals provide a **complete basis** for all one-loop gluon amplitudes in $\mathcal{N} = 4$ [Bern, Dixon, Kosower].

$$\mathcal{A}^{1\text{-loop}} = \sum_{\text{boxes}} (\text{coefficient})$$



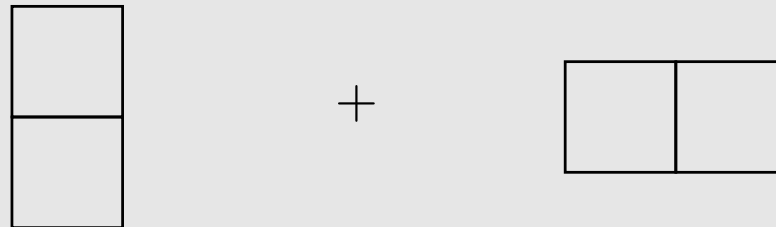
Unitarity methods can be used to determine the **coefficients** for a desired amplitude [Britto, Cachazo, Feng].

Higher Loops

Unitarity based methods for computing the coefficients can be generalized to higher loop amplitudes [Cachazo, Buchbinder] [Bern, Dixon, Smirnov] [Bern, Carrasco, Johansson, Kosower]

The problem is that the complete basis of integrals is not known even for all two-loop amplitudes!

For example, the two-loop four-particle amplitude is given by the sum of only two scalar integrals



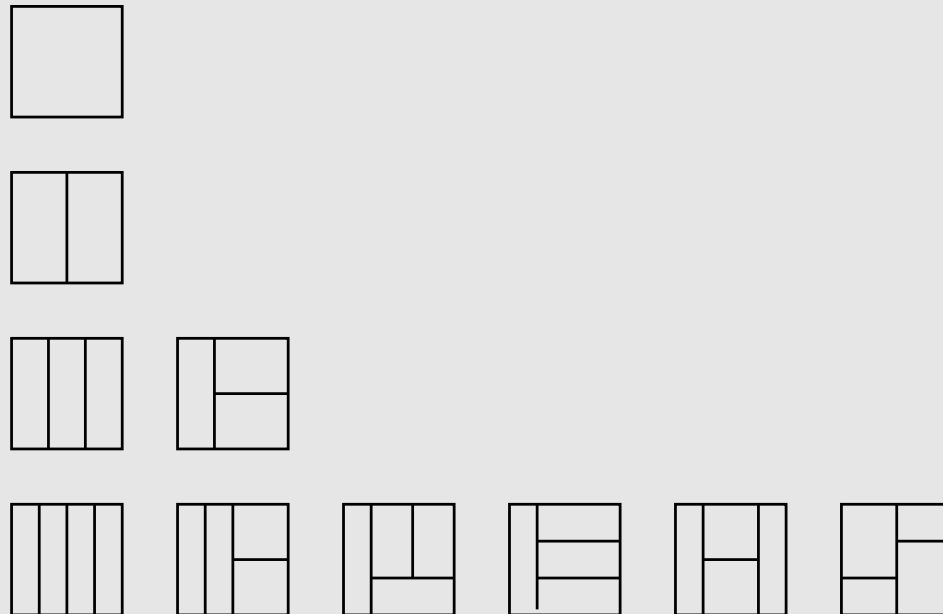
But in general it is very difficult to determine which integrals contribute to any particular amplitude.

$\mathcal{N} = 4$ Yang-Mills Status Report



The Rung Rule

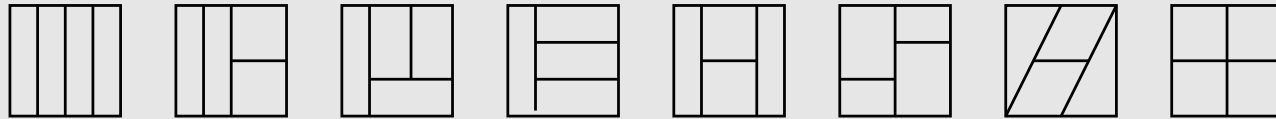
Cuts involving only two particles in an intermediate channel are particularly easy to analyze; and in fact the relevant algebra extrapolates nicely to all loops—this analysis led to what is called the ‘rung rule’ [Bern, Rozowsky, Yan] which is easiest to explain in pictures.



At two and three loops, this is all there is.

Rung Rule=Wrong Rule

However at four loops the amplitude contains two additional integrals.



[Bern, Czakon, Dixon, Kosower, Smirnov]

The integrand of the five-loop amplitude involves the 24 rung rule integrals and an additional 10 non-rung-rule contributions. [Bern, Carrasco, Johansson, Kosower]

It seems that non-rung rule contributions can be determined by 'dual conformality'. [Drummond, Henn, Smirnov, Sokachev]

The rest of my talk will be about the difficult problem of **evaluating** integrals!

Iterative Relations

Based on resummation work by **Sterman and Tejeda-Yeomans**, and infrared singularity work by **Catani**, it is believed that the planar n -particle L -loop MHV amplitude satisfies an iterative relation of the form

$$\mathbf{M}_n^{(L)}(\epsilon) = \mathbf{P}^{(L)}(\mathbf{M}_n^{(1)}(\epsilon), \dots, \mathbf{M}_n^{(L-1)}(\epsilon)) + \mathbf{f}^{(L)}(\epsilon)\mathbf{M}_n^{(1)}(L\epsilon) + \mathbf{C}^{(L)} + \mathcal{O}(\epsilon),$$

where $P^{(L)}$ are some known polynomials.

The only ‘unknown’ quantities in iterative relations are $f^{(L)}(\epsilon)$ and $C^{(L)}$.

At weak coupling perturbatively verified for $(L, n) = (2, 4), (3, 4), (2, 5)$
[Anastasiou, Bern, Dixon, Kosower] [Bern, Dixon, Smirnov] [Cachazo, Spradlin, AV] [Bern, Czakon, Kosower, Roiban, Smirnov].

At strong coupling **Alday and Maldacena** gave a prescription for computing gluon scattering amplitudes using AdS/CFT and found agreement with the iterative ansatz for $n = 4$ and predicted they break down at $n = \infty!$

Iterative Relations and Anomalous Dimension

The iterative relations imply that one has to sift all the way through to the ϵ^{-2} in order to find any 'new' information—the vast majority of the rational coefficients which specify the L -loop amplitude are completely determined in terms of lower loop amplitudes.

$$M^{(L)} = \frac{\text{fixed}}{\epsilon^{2L}} + \dots + \frac{\text{fixed}}{\epsilon^3} + \frac{\text{fixed} + 1 \text{ unfixed number}}{\epsilon^2} + \frac{\text{fixed} + 1 \text{ unfixed number}}{\epsilon} + \left[\text{fixed} + 2 \text{ unfixed numbers} \right] + \mathcal{O}(\epsilon). \quad (1)$$

The one unfixed number at order ϵ^{-2} is the L -loop contribution to the **cusplike anomalous dimension** $f(\lambda)$.

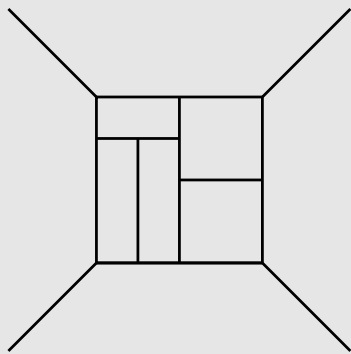
The one unfixed number at order ϵ^{-1} is the L -loop contribution to the **collinear anomalous dimension** $g(\lambda)$.

**It would be nice to develop some kind of technology which would help us
isolate just anomalous dimension.**

Some New Loop Technology

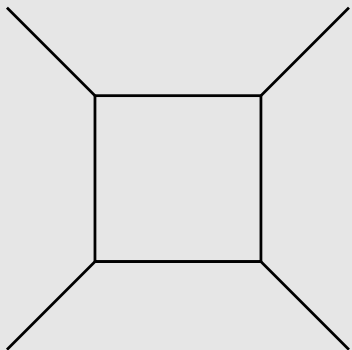
STEP 1. We observe that any dimensionally regulated L -loop four-particle Feynman integral can be written in the form (Mellin-Barnes representation)

[Smirnov, Tausk, Czakon, ...]



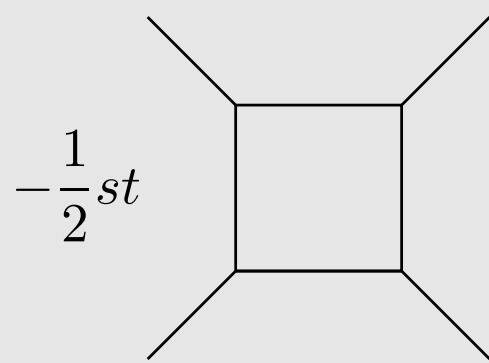
$$= \int_{-i\infty}^{+i\infty} dy x^y F(y, \epsilon), \quad F(y, \epsilon) = F(-y, \epsilon), \quad x = s/t$$

for some function $F(y, \epsilon)$, which is relatively easy to determine. As an example (not representative, because of its simplicity):



$$\implies F(y, \epsilon) = \Gamma(1 + \frac{1}{2}\epsilon + y) \Gamma^2(y - \frac{1}{2}\epsilon) \Gamma^2(-y - \frac{1}{2}\epsilon) \Gamma(1 - \frac{1}{2}\epsilon - y).$$

The final integral over y is the really nasty one.



Integral over y

$$\begin{aligned}
 -\frac{1}{2}st &= -\frac{2}{\epsilon^2} + \frac{1}{4}L^2 + \frac{2\pi^2}{3} + \epsilon \left[-H_{001}(x) - LH_{01}(x) \right. \\
 &\quad \left. -\frac{1}{2}L^2H_1(x) - \frac{\pi^2}{2}H_1(x) + \frac{11\pi^2}{12}L + \frac{17\zeta(3)}{3} \right] \\
 &+ \epsilon^2 \left[H_{0001}(x) + H_{0011}(x) + H_{0101}(x) + H_{1001}(x) - \frac{1}{2}LH_{001}(x) \right. \\
 &\quad \left. -LH_{011}(x) - LH_{101}(x) + \frac{1}{2}L^2H_{11}(x) + \frac{\pi^2}{2}H_{11}(x) + \frac{1}{12}L^3H_1(x) \right. \\
 &\quad \left. -\zeta(3)H_1(x) + \frac{\pi^2}{4}LH_1(x) + \frac{1}{64}L^4 + \frac{\pi^2}{24}L^2 - \frac{\zeta(3)}{2}L + \frac{41\pi^4}{720} \right] \\
 &+ \epsilon^3 [37 \text{ terms}] + \epsilon^4 [79 \text{ terms}] + \mathcal{O}(\epsilon^5)
 \end{aligned}$$

where $L = \ln(x)$ and $x = t/s$.

Adapted from [\[Bern, Dixon, Smirnov\]](#).

Some New Loop Technology

STEP 1. Any four-particle integral $= \int dy x^y F(y, \epsilon)$.

STEP 2. If we want to study some iterative equation, it is clearly tempting to try to collect all of the terms appearing in some relation inside one y integral, and then expand through $\mathcal{O}(\epsilon)$ **under the y integral**.

This is not possible, because $F(y, \epsilon)$ has poles which collide with the integration contour $\text{Re}(y) = 0$ at $\epsilon = 0$, e.g.

$$F(y, \epsilon) = \Gamma(1 + \frac{1}{2}\epsilon + y)\Gamma^2(y - \frac{1}{2}\epsilon)\Gamma^2(-y - \frac{1}{2}\epsilon)\Gamma(1 - \frac{1}{2}\epsilon - y).$$

This signals that expanding in ϵ and performing the y integral **do not commute**—we are not allowed to expand in ϵ under the integral. We call these annoying poles **obstructions** because they **obstruct** our ability to collect everything under a single integral which is valid near $\epsilon = 0$.

Obstructions

We can uniquely decompose any amplitude into

$$\int_{-i\infty}^{+i\infty} dy x^y [\text{nonsingular}(y) + \text{obstruction}(y)],$$

where the obstruction terms are given by a polynomial in $\ln^2 x$.

Obstructions have a product algebra structure and must separately satisfy any iterative relation.

In order to read off the L -loop cusp anomalous dimension from an L -loop four-gluon amplitude, it is sufficient to start with the (relatively far simpler) expressions for the **Mellin transform** of the amplitude, and then just read off the obstruction since only this particular coefficient contributes to the cusp anomalous dimension. **[Cachazo, Spradlin, AV]**

This algorithm is easily implemented in Mathematica (building on very nice code written by **Czakon**), and greatly optimizes the calculation of the cusp anomalous dimension.

Example: three-loops

After throwing away all of the 'bulk terms', one can set $x = 1$, which is equivalent to keeping only the constant term in each polynomial $P(\ln^2 x)$.

The quantities

$$M^{(1)}(\epsilon) \sim -\frac{2}{\epsilon^2} + \frac{2\pi^2}{3} + \epsilon \frac{17\zeta(3)}{3} + \epsilon^2 \frac{41\pi^4}{720} + \mathcal{O}(\epsilon^3),$$

$$M^{(2)}(\epsilon) \sim \frac{2}{\epsilon^4} - \frac{1}{\epsilon^2} \frac{5\pi^2}{4} - \frac{1}{\epsilon} \frac{65\zeta(3)}{6} - \frac{\pi^4}{90} + \mathcal{O}(\epsilon),$$

$$M^{(3)}(\epsilon) \sim -\frac{4}{3\epsilon^6} + \frac{1}{\epsilon^4} \frac{7\pi^2}{6} + \frac{1}{\epsilon^3} \frac{31\zeta(3)}{3} - \frac{1}{\epsilon^2} \frac{161\pi^4}{3240} + \mathcal{O}(\epsilon^{-1}).$$

obey the expected relation

$$M^{(3)}(\epsilon) = -\frac{1}{3} (M^{(1)}(\epsilon))^3 + M^{(1)}(\epsilon)M^{(2)}(\epsilon) + \frac{11\pi^4}{180} M^{(1)}(3\epsilon) + \mathcal{O}(1/\epsilon).$$

Four-Loop Cusp Anomalous Dimension

Using our method we obtained [Cachazo, Spradlin, AV]

$$f^{(4)} = -117.1789 \pm 0.0002$$

in very good agreement with the **BES/BHL** conjecture

$$f^{(4)} = -(4\zeta(2)^3 + 24\zeta(2)\zeta(4) + 50\zeta(6) + 4\zeta(3)^2) = -117.1788285\dots$$

and a factor of 1000 improvement over the **BCDKS** result

$$f^{(4)} = -117.2 \pm 0.2$$

The improvement is possible because we only need to compute the ‘obstructions’, not the full amplitude (throw most of the terms away).

Comment: If we make an ansatz

$$f^{(4)} = -(n_1\zeta(2)^3 + n_2\zeta(2)\zeta(4) + n_3\zeta(6) + n_4\zeta(3)^2)$$

where n_i are **integers** then there is only one choice of n_i .

Four-loop Collinear Anomalous Dimension

Perturbative calculations through three loops have established that collinear anomalous dimension [Bern, Dixon, Smirnov]

$$g(\lambda) = -4\zeta_3 \left(\frac{\lambda}{16\pi^2} \right)^2 + 8 \left(4\zeta_5 + \frac{5}{9}\pi^2\zeta_3 \right) \left(\frac{\lambda}{16\pi^2} \right)^3 + g^{(4)} \left(\frac{\lambda}{16\pi^2} \right)^4 + \mathcal{O}(\lambda^5).$$

At strong coupling Alday and Maldacena predict

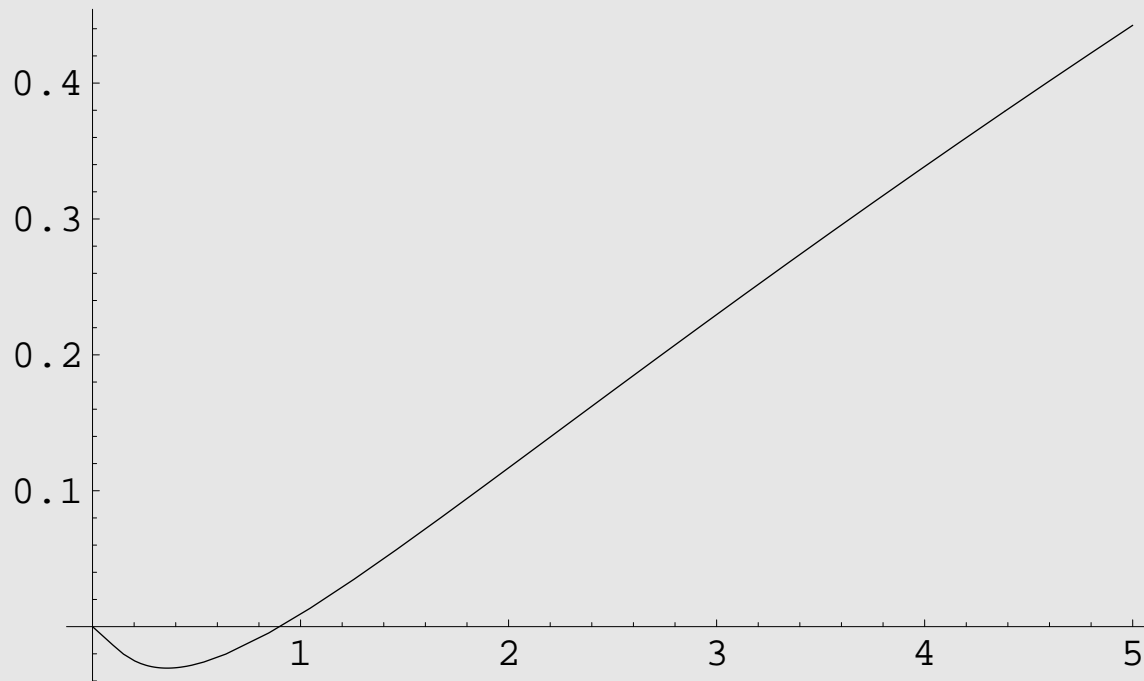
$$g(\lambda) = 2(1 - \log 2) \sqrt{\frac{\lambda}{16\pi^2}} + \mathcal{O}(1).$$

Using our method we find [Cachazo, Spradlin, AV]

$$g^{(4)} = -1240.9 \pm 0.3.$$

A Guess for What $g(\lambda)$ Might Look Like

A Padé approximant for $g(\lambda)$ based upon all available data (through four loops at weak coupling and leading order at strong coupling) looks like



Relate $g(\lambda)$ to observable computed by integrability technique? Integral equation for $g(\lambda)$?

Dual Conformally Invariant Off-Shell Integrals

Evidence has recently emerged for a hidden symmetry of scattering amplitudes in $\mathcal{N} = 4$ Yang-Mills called **dual conformal symmetry**. [Drumond, Korchemsky, Sokatchev]

We classified and evaluated dual conformally invariant **off-shell** four-point diagrams through four loops. Found two new magic identities. [Nguyen, Spradlin, AV]

Off-shell dual conformal integrals are vastly simpler to study: they are finite in four dimensions, one term Mellin-Barnes representation, ladder diagrams has been obtained for L-loops several years ago, they satisfy magic identities.

Drawback to working off-shell: we don't know which integrals contribute to the off-shell amplitude. If the iterations could be made to work with off-shell integrals, it would provide a much simpler method for computing the cusp anomalous dimension!

Open Questions

How far need one calculate before unlocking all the structure?

How much is gained by adding one more loop, or one more leg?

In the case of loops, there were strong reasons to suspect that special things would start happening at four loops (and they did!) so there was great interest in the calculation of the four-loop cusp anomalous dimension.

The case for pushing that to five loops is comparatively weak. (Cancellation of $\zeta(6, 2)$?)

The next frontier is clearly to add legs, since Alday-Maldacena suggests that the BDS ansatz might break down starting at $n = 6$ particles, and to uncover the mysterious role of dual conformal symmetry.

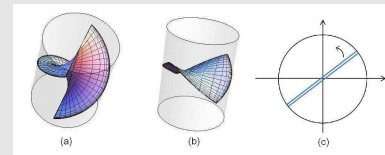
Historically we have learned a lot from studying alleged 'breakdowns' of AdS/CFT. Of course it has never been AdS/CFT that has broken down, only our naivety.

On iterations beyond four gluons

On the **gauge theory** side: we are in the process of doing a perturbative computation for **six gluons at two loop** to make a verdict on BDS ansatz. [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, AV, in progress]

On the **string theory** side: cleverness is required to find string solution for five and more gluons. Use integrability?

We used a Pohlmeyer type reduction to **generate classical string solutions in AdS**. Null cusp solution is sinh-Gordon vacuum. Constructed string solutions corresponding to sinh-Gordon solitons and breathers by inverse scattering technique and also by sigma model dressing. [Jevicki, Jin, Kalousios, AV]



Conclusion

We developed some techniques to aid in direct tests of the conjectured planar $\mathcal{N} = 4$ Yang-Mills S-matrix and multiloop iterative relations. As an application, we computed four-loop cusp and collinear anomalous dimensions.

The motivation behind this research is the desire to explore and uncover the rich mathematical structure underlying $\mathcal{N} = 4$ Yang-Mills theory.

Discovering such structures also has the pleasant side benefit of making previously difficult calculations much simpler.

Prospects are great for continued progress, both in supersymmetric gauge theories as well as QCD. There is definitely a lot more to learn and discover.