Towards understanding superstring theory in $AdS_5 \times S^5$

- Review of some recent progress

  including 2-loop superstring results

  R. Roiban and A. T., arXiv:0709.0681

- Reformulation of $AdS_5 \times S^5$ superstring in terms of current variables: "Pohlmeyer reduction"

  M. Grigoriev and A. T., arXiv:0711.0155

AdS/CFT

$\mathcal{N} = 4$ SYM at $N = \infty$

dual to type IIB superstrings in $AdS_5 \times S^5$

$\lambda = g^2_{_{YM}} N$ related to string tension

$2\pi T = \frac{R^2}{\alpha'} = \sqrt{\lambda}$

$g_s = \frac{\lambda}{4\pi N} \to 0$

need to go beyond BPS states and
"supergravity + classical probes" approximation

recent remarkable progress in quantitative understanding of
interpolation from weak to strong 't Hooft coupling
based on interplay of quantum gauge and string results
and assumption of exact integrability

string energies = dimensions of gauge-invariant operators

$$E(\sqrt{\lambda}, J, m, ...) = \Delta(\lambda, J, m, ...)$$

J - charges of $SO(2,4) \times SO(6)$:

spins $S_1, S_2;\ J_1, J_2, J_3$

m - windings, folds, cusps, oscillation numbers, ...

Operators: $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3}D_+^{S_1}D_\perp^{S_2}...F_{mn}...\Psi...)$

Solve susy 4-d CFT = string in R-R background:

compute $E = \Delta$ for any λ (and J,m)

Perturbative expansions are opposite:

$\lambda \gg 1$ in perturbative string theory

$\lambda \ll 1$ in perturbative planar gauge theory

use perturbative results on both sides

and other properties (integrability, susy,...)

to come up with an exact answer (Bethe ansatz,...)

Last 5 years: remarkable progress:

"semiclassical" string states with large quantum numbers

dual to "long" gauge operators (BMN, GKP, ...)

$E = \Delta$ – same dependence on $J, m, ...$

coefficients = interpolating functions of λ

SYM: dilatation operator that determines Δ

is same as an integrable spin chain Hamiltonian

integrability at both perturbative gauge ($\lambda \ll 1$)

and string ($\lambda \gg 1$) sides

suggests Bethe ansatz for the spectrum at any λ

$$e^{ip_k J} = \prod_{j \neq k}^{M} S(p_k, p_j; \lambda), \qquad S = S_1\, e^{i\theta}$$

$$S_1 = \frac{u_k - u_j + i}{u_k - u_j - i}\,, \qquad \theta = \theta(p_k, p_j; \lambda)$$

scattering of elementary excitations (magnons) with 1-d momenta p_j and rapidities u_j

$u_j(p_j, \lambda) = \frac{1}{2} \cot \frac{p_j}{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}}$

$E = J + \sum_{j=1}^{M} \left(\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}} - 1 \right)$

What about phase θ ?

structure fixed by symmetries (Beisert 05)

$$\theta(p, p'; \lambda) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{rs}(\lambda) \left[q_s(p') q_r(p) - q_s(p) q_r(p') \right]$$

$$q_{r+1}(p) = \frac{2}{r} \sin \frac{rp}{2} \left(\frac{\sqrt{1+\frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} - 1}{\frac{\lambda}{\pi^2} \sin \frac{p}{2}} \right)^r ,$$

$c_{rs}(\lambda) = ?$

crucial imput from string theory:

$c_{rs}(\lambda \gg 1) = \lambda^{\frac{r+s-1}{2}} \left[\delta_{r,s-1} + \frac{1}{\sqrt{\lambda}} a_{rs} + \frac{1}{(\sqrt{\lambda})^2} b_{rs} + ... \right]$

String 1-loop corrections to string energies

(Frolov, AT 03; Park, Tirziu, AT 05) $\to a_{rs} \neq 0$ (Beisert, AT 05)

1-loop string results translate into (Hernandez, Lopez 06)

$a_{rs} = \frac{2}{\pi}[1-(-1)^{r+s}]\frac{(r-1)(s-1)}{(r-1)^2-(s-1)^2}$

Consistent (Arutyunov, Frolov 06; Beisert 06)

with “crossing” (Janik 06)

All-order guess for strong coupling expansion

(Beisert, Hernandez, Lopez 06)

A year ago finally fixed completely (Beisert, Eden, Staudacher 06)

comparing to weak-coupling results (4-loop result of Bern et al)

But first-principles derivation remains to be given

Problem:

solve string theory in $AdS_5 \times S^5$ –

determine the magnon (BMN excitation) scattering S-matrix

String Theory in $AdS_5 \times S^5$

bosonic coset $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

generalized to supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ (Metsaev, AT 98)

$$S = T \int d^2\sigma \Big[G_{mn}(x)\partial x^m \partial x^n + \bar{\theta}(D + F_5)\theta\partial x$$

$$+ \bar{\theta}\theta\bar{\theta}\theta\partial x \partial x + ... \Big]$$

tension $T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical integrability of coset σ-model (Luscher-Pohlmeyer 76)

same for $AdS_5 \times S^5$ superstring (Bena, Polchinski, Roiban 02)

Progress in understanding of implications of (semi)classical integrability (Kazakov, Marshakov, Minahan, Zarembo 04; Beisert et al 05; Dorey, Vicedo 06,...)

Explicit computation of 1-loop quantum superstring corrections to classical string energies (Frolov, AT 02-4, ...)
results were used as input for 1-loop term
in strong-coupling expansion of the phase θ in BA

This year:
2-loop string corrections (Roiban, Tirziu, AT; Roiban, AT 07)
2-loop check of finiteness of the GS superstring;
agreement with BA
– implicit check of integrability of quantum string theory
– non-trivial confirmation of BES exact phase in BA

Universal scaling function = Cusp anomalous dimension

gauge theory: $\mathrm{Tr}(\Phi D_+^S \Phi)$

$$\Delta = S + 2 + f(\lambda)\ln S + ..., \qquad S \gg 1$$

$$f(\lambda \ll 1) = c_1\lambda + c_2\lambda^2 + c_3\lambda^3 + c_4\lambda^4 + ...$$

c_n are given by Feynmann graphs of 4d CFT – N=4 SYM

string theory: GKP folded string with spin S in AdS_5

$$f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi}[a_0 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{(\sqrt{\lambda})^2} + ...]$$

a_n are given by Feynmann graphs of 2d CFT – $AdS_5 \times S^5$ string

Explicitly:

$$f_{\lambda \ll 1} = \frac{1}{2\pi^2}\Big[\lambda - \frac{\lambda^2}{48} + \frac{11\lambda^3}{2^8 \times 45} - (\frac{73}{630} + \frac{4(\zeta(3))^2}{\pi^6})\frac{\lambda^4}{2^7} + ...\Big]$$

c_3: Kotikov, Lipatov, et al 03; c_4: Bern, Dixon, et al 06

$$f_{\lambda \gg 1} = \frac{\sqrt{\lambda}}{\pi}[1 - \frac{3\log 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} + ...]$$

a_0: Gubser, Klebanov, Polyakov 02;

a_1: Frolov, AT 02

a_2: Roiban, AT 07

$K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.915...$ – Catalan's constant

appears from 2-loop sigma model integrals

Smooth interpolation from weak to strong coupling

Remarkably, both expansions are reproduced from single
Beisert-Eden-Staudacher integral equation for $f(\lambda)$
obtained using the exact BES phase in the BA

strong-coupling expansion of BES eq.:
numerical results: Benna, Benvenutti, Klebanov, Scardicchio 07
analytic expansion: Basso, Korchemsky, Kotansky 07
in full agreement with string theory
$\rightarrow$ 2-loop string check of the BES phase

BES equation: $f(\lambda)$ known at least in principle to
any order in small λ and large λ expansion
Exact solution hopefully will be can be found soon...

weak coupling (BES): $f(\lambda) = \sum_{n=1}^{\infty} c_n (\frac{\lambda}{4\pi^2})^n$

$$c_1 = 2, \quad c_2 = -\zeta_2, \quad c_3 = 88\zeta_4\,, \quad c_4 = -16(73\zeta_6 + 4\zeta_3^2)$$

$$c_5 = 32(887\zeta_8 + 8\zeta_2\zeta_3^2 + 40\zeta_3\zeta_5)$$

$$c_6 = -64(136883\zeta_{10} + 8\zeta_2\zeta_3^2 + 80\zeta_2\zeta_3\zeta_5 + 210\zeta_3\zeta_7 + 102\zeta_5^2), \quad \ldots$$

strong coupling (BKK): $f(\lambda) = \frac{\sqrt{\lambda}}{\pi} \sum_{n=0}^{\infty} \frac{a_n}{(\sqrt{\lambda})^n}$

$$a_0 = 1, \quad a_1 = -3\zeta_1, \quad a_2 = -\beta_2\,, \quad a_3 = -\frac{1}{32}\left[27\zeta_3 + 96\beta_2\beta_1\right]$$

$$a_4 = -\frac{1}{16}\left[84\beta_4 + 81\zeta_3\beta_1 + 32\beta_2^2 + 144\beta_2\beta_1^2\right]\,,$$

$$a_5 = -\frac{9}{2048}\left[4785\zeta_5 + 10572\beta_4\zeta_1 + 4416\zeta_3\beta_2 + 5184\zeta_3\zeta_1^2 + 4096\beta_2^2\zeta_1\right], \quad \ldots$$

$$\zeta_k = \sum_{n=1}^{\infty} \frac{1}{n^k}\,, \quad \zeta_{2n} \sim \pi^{2n}, \quad \beta_1 \equiv -\sum \frac{(-1)^n}{n} = \ln 2\,,$$

$$\beta_k = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^k}\,, \quad \beta_2 = \mathrm{K}\,, \quad \beta_{2n+1} \sim \pi^{2n+1}$$

non-trivial function in 4d QFT

Beyond 2-loop order in string theory ?

Deeper understanding of quantum string theory
from integrability point of view?

Exact string S-matrix?

Proof of the BES Bethe ansatz ?

How to solve quantum string theory in $AdS_5 \times S^5$?

GS string on supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$
not of known solvable type (cf. free oscillators; WZW)
analogy with exact solution of $O(n)$ model (Zamolodchikovs) or
principal chiral model (Polyakov-Wiegmann; KWZ; ...) ?
– 2d CFT – no mass generation

Try as in flat space –
light-cone gauge: analog of $x^+ = p^+\tau,\ p^+ = \text{const},\ \Gamma^+\theta = 0$
Two natural options:
(i) null geodesic parallel to the boundary in Poincare patch –
action/Hamiltonian quartic in fermions (Metsaev, Thorn, AT, 01)
(ii) null geodesic wrapping S^5:
hidden $su(2|2) \times su(2|2)$ symmetry
but complicated action (Callan et al, 03;
Arutyunov, Frolov, Plefka, Zamaklar, 05-06)

Common problem:

lack of manifest 2d Lorentz symmetry

hard to apply known 2d integrable field theory methods –
S-matrix depends on two rapidities, not on their difference only
constraints on it are unclear, etc.

An alternative approach: “Pohlmeyer reduction”
use conf. gauge, solve Virasoro conditions in terms of currents,
find “reduced” action for physical number of d.o.f.,
use it as a starting point for quantization

compare to two related models:

I. “non-abelian dual” for PCM
(Zakharov, Mikhailov 78; Nappi 80)
– solve EOM’s in terms of currents,
consider flatness condition (MC) as dynamical

$$L = \mathrm{Tr}(J_a J^a) , \qquad J_a = g^{-1}\partial_a g$$

$$\partial_a J^a = 0\ , \qquad \partial_a J_b - \partial_b J_a + [J_a, J_b] = 0$$

Solve EOM by $J_a = \epsilon_{ab}\partial^b\chi, \ \ \chi \in \mathfrak{g}$

then from flatness (MC)

$$\partial^a\partial_a\chi - \epsilon^{ab}\partial_a\chi\partial_b\chi = 0$$

following from

$$L = \mathrm{Tr}(\partial^a\chi\partial_a\chi + \frac{2}{3}\epsilon^{ab}\chi[\partial_a\chi, \partial_b\chi])$$

corresponds to a gauge-equivalent choice of classical

Lax pair (Mikhailov-Zakharov 78)

But: does not solve Virasoro conditions;

does not define equivalent quantum theory

(Nappi; Fridling, Jevicki 84; Fradkin, AT 85)

Another attempt:

II. FR model (Faddeev, Reshetikhin 86)

express PCM + Virasoro in terms of two

constrained currents as basic variables

fix conf. symm. or add Virasoro for $R_t \times G$ ($X^0 = \mu\tau$)

$$\mathrm{Tr}(J_+ J_+) = \mu^2 \ , \qquad \mathrm{Tr}(J_- J_-) = \mu^2$$

in addition to EOM combined with MC into

$$D_- J_+ = 0 \ , \qquad D_+ J_- = 0 \ , \qquad D_a = \partial_a + [J_a, \ ]$$

e.g. $G = S^3 = SU(2)$: take $n^i_\pm = \mu^{-1} J^i_\pm$

as two unit vectors to solve Virasoro; action:

$$S = \int d^2\sigma [C_+(n_-) + C_-(n_+) + \mu^2 n^i_+ n^i_-] \ ,$$

where $C_a(J^i) \equiv -\frac{1}{2} \int_0^1 dy \ \epsilon_{ijk} n^i \partial_a n^j \partial_y n^k$

get first-order action for 2+2=4 independent d.o.f.

But: 2d Lorentz invariance is missing – broken by constraints

Remarkably, there is an alternative system with standard
2d Lorentz invariant second-order action
for 2 dynamical d.o.f. (1+3-2=2)
describing the same $R_t \times S^3$ string equations of motion
Complex sine-Gordon model found by Pohlmeyer reduction (PR)

$$\widetilde{S} = \int d^2\sigma [\partial_+\varphi\partial_-\varphi + \tan^2\varphi\, \partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi]$$

CSG: an example of non-abelian Toda model (Leznov, Saveliev):
related to massive integrable perturbation of a coset WZW model
–here $SO(3)/SO(2)$ (Hollowood, Miramontes, Park 94)
quantum-integrable: S-matrix is known (Dorey, Hollowood 95)

Aim: construct PR version for $AdS_5 \times S^5$ superstring

(i) introduce new fields locally related to supercoset currents

(ii) solve conformal gauge (Virasoro) condition explicitly

(iii) find local 2d Lorentz-invariant

action for independent (8B+8F) d.o.f

– fermionic generalization of non-abelian Toda theory

PR: a nonlocal map that preserves integrable structure

1. gauge-equivalent Lax pairs; map between soliton solutions

gives integrable massive local field theory

2. quantum equivalence to original GS model ?

may expect for full $AdS_5 \times S^5$ string model = CFT

3. integrable theory: semiclassical solitonic spectrum

may essentially determine quantum spectrum

the two solitonic S-matrices should be closely related:

Lorentz-invariant S-matrix of PR-model should effectively

give the complicated magnon S-matrix

Pohlmeyer reduction: bosonic coset models

Prototypical example: S^2-sigma model $\rightarrow$ Sine-Gordon theory

$$L = \partial_+ X^m \partial_- X^m - \Lambda(X^m X^m - 1)\ , \qquad m = 1, 2, 3$$

Equations of motion:

$$\partial_+\partial_- X^m + \Lambda X^m = 0\,, \quad \Lambda = \partial_+ X^m \partial_- X^m\,, \quad X^m X^m = 1$$

Stress tensor: $\mathrm{T}_{\pm\pm} = \partial_\pm X^m \partial_\pm X^m$

$$\mathrm{T}_{+-} = 0\ , \quad \partial_+ \mathrm{T}_{--} = 0\,, \qquad \partial_- \mathrm{T}_{++} = 0$$

implies $\mathrm{T}_{++} = f(\sigma_+), \mathrm{T}_{--} = h(\sigma_-)$

using the conformal transformations $\sigma_\pm \rightarrow F_\pm(\sigma_\pm)$ can set

$$\partial_+ X^m \partial_+ X^m = \mu^2\ , \qquad \partial_- X^m \partial_- X^m = \mu^2\,, \qquad \mu = \text{const}\ .$$

3 unit vectors in 3-dimensional Euclidean space:

$$X^m\,, \qquad X_+^m = \mu^{-1}\partial_+ X^m\,, \qquad X_-^m = \mu^{-1}\partial_- X^m\,,$$

X^m is orthogonal ($X^m \partial_\pm X^m = 0$) to both X_+^m and X_-^m
remaining $SO(3)$ invariant quantity is scalar product

$$\partial_+ X^m \partial_- X^m = \mu^2 \cos 2\varphi$$

then $\partial_+ \partial_- \varphi + \frac{\mu^2}{2} \sin 2\varphi = 0$
following from sine-Gordon action (Pohlmeyer, 1976)

$$\widetilde{L} = \partial_+ \varphi \partial_- \varphi + \frac{\mu^2}{2} \cos 2\varphi$$

2d Lorentz invariant despite explicit constraints
Classical solutions and integrable structures
(Lax pair, Backlund transformations, etc) are directly related
e.g., SG soliton mapped into rotating folded string on S^2
"giant magnon" in the $J = \infty$ limit (Hofman, Maldacena 06)
other examples for CSG (Chen, Dorey, Okamura 06;
Okamura, Suzuki, Hayashi, Vicedo 07;
Jevicki, Spradlin, Volovich, et al 07)

Analogous construction for S^3 model gives

Complex sine-Gordon model (Pohlmeyer; Lund, Regge 76)

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \tan^2\varphi\,\partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi$$

φ, θ are $SO(4)$-invariants:

$$\mu^2 \cos 2\varphi = \partial_+ X^m \partial_- X^m$$

$$\mu^3 \sin^2\varphi\,\partial_\pm\theta = \mp\frac{1}{2}\epsilon_{mnkl} X^m \partial_+ X^n \partial_- X^k \partial_\pm^2 X^l$$

"String on $R_t \times S^n$" interpretation

conformal gauge plus $t = \mu\tau$ to fix conformal diffeomorphisms:

$\partial_\pm X^m \partial_\pm X^m = \mu^2$ are Virasoro constraints

Similar construction for AdS_n case,

i.e. string on $AdS_n \times S^1_\psi$ with $\psi = \mu\tau$

e.g. reduced theory for $AdS_3 \times S^1$

$$\widetilde{L} = \partial_+\phi\partial_-\phi + \tanh^2\varphi\,\partial_+\chi\partial_-\chi - \frac{\mu^2}{2}\cosh 2\phi$$

Comments:

- Virasoro constraints are solved by a special choice of variables related nonlocally to the original coordinates
- Although the reduction is not explicitly Lorentz invariant the resulting Lagrangian turns out to be 2d Lorentz invariant
- The reduced theory is formulated in terms of manifestly $SO(n)$ invariant variables: "blind" to original global symmetry
- The reduced theory is equivalent to the original theory as an integrable system: the respective Lax pairs are gauge-equivalent
- PR may be thought of as a formulation in terms of physical d.o.f. – coset space analog of flat-space l.c. gauge (where 2d Lorentz is unbroken)
- In general reduced theory can not be quantum-equivalent to the original one (e.g., conformal symmetry was assumed in the reduction procedure)

PR for bosonic F/G-coset model

To find reduced theory for $AdS_5 \times S^5$ GS model need to understand PR of F/G coset sigma models as G/H gauged WZW models modified by relevant integrable potential and then generalize to GS supercoset

F/G-coset sigma model:

symmetric space condition ($\mathfrak{f}$, $\mathfrak{g}$ are Lie algebras of F and G)

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g} \,, \qquad [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \,, \qquad [\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p} \,, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g}$$

with $\langle \mathfrak{g}, \mathfrak{p} \rangle = 0$ (choose $\langle a, b \rangle = \mathrm{Tr}(ab)$)

Lagrangian:

$$L = -\mathrm{Tr}(P_+ P_-) \,, \qquad P_\pm = (f^{-1} \partial_\pm f)_{\mathfrak{p}} \,,$$

$$J = f^{-1} df = \mathcal{A} + P \,, \qquad \mathcal{A} = J_{\mathfrak{g}} \in \mathfrak{g} \,, \qquad P = J_{\mathfrak{p}} \in \mathfrak{p} \,.$$

Symmetries: G gauge transformations $f \to fg$;

global F-symmetry: $f \to f_0 f$, $f_0 = \mathrm{const} \in F$

classical conformal invariance

Equations of motion in terms of currents

let now $J = \mathcal{A} + P$ be fundamental variables, not f

$$D_+ P_- = 0\,, \quad D_- P_+ = 0\,, \qquad D = d + [\mathcal{A},\ ] \qquad \text{– EOM}$$

$$D_- P_+ - D_+ P_- + [P_+, P_-] + [A_-, A_+] = 0 \qquad \text{– Maurer-Cartan}$$

$$\mathrm{Tr}(P_+ P_+) = -\mu^2\,, \quad \mathrm{Tr}(P_- P_-) = -\mu^2 \qquad \text{– Virasoro}$$

Main idea: – first solve EOM and Virasoro and then MC

using special choice of G gauge condition and conformal diffs

then find reduced action giving eqs. resulting from MC

gauge fixing that solves the first Virasoro constraint

$$P_+ = \mu\, T = \text{const}\,, \qquad T \in \mathfrak{p} = \mathfrak{f} \ominus \mathfrak{g}, \qquad \mathrm{Tr}(TT) = -1$$

choice of special element $T \to$ decomposition of the algebra of F

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g}\,, \qquad \mathfrak{p} = T \oplus \mathfrak{n}\,, \qquad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}\,, \qquad [T, \mathfrak{h}] = 0\,,$$

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}\,, \qquad [\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}\,, \qquad [T, \mathfrak{m}] \subset \mathfrak{n}\,, \qquad [T, \mathfrak{n}] \subset \mathfrak{m}\,.$$

$\mathfrak{h}$ is a centraliser of T in $\mathfrak{g}$

EOM $D_- P_+ = 0$ is solved by

$$(\mathcal{A}_-)_{\mathfrak{m}} = 0\,, \qquad \mathcal{A}_- = (\mathcal{A}_-)_{\mathfrak{h}} \equiv A_-$$

second Virasoro constraint is solved by

$$P_- = \mu\, g^{-1} T g\,, \qquad\qquad g \in G$$

EOM $D_+ P_- = 0$ is solved by

$$\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1} A_+ g$$

To summarise:

we solved EOM's and Virasoro constraints introducing
new dynamical field variables

G-valued field g , $\qquad$ $\mathfrak{h}$-valued fields $\quad A_+,\ A_-,\quad [T, A_\pm] = 0$

what remains is the Maurer-Cartan equation

Relation to G/H gauged WZW model

Maurer-Cartan equation in terms of new parametrization:

$$\partial_-(g^{-1}\partial_+ g + g^{-1}A_+ g) - \partial_+ A_- \\ + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g] + \mu^2[g^{-1}Tg, T] = 0$$

Recall: $J = f^{-1}df = \mathcal{A} + P,\ P_+ = \mu T,\ P_- = \mu g^{-1}Tg$

$\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1}A_+ g, \quad \mathcal{A}_- = A_-$

MC eq. has "on-shell" $H \times H$ gauge symmetry:

$$g \to h^{-1} g \bar{h}\,,$$

$$A_+ \to h^{-1}A_+ h + h^{-1}\partial_+ h\,, \quad A_- \to \bar{h}^{-1}A_- \bar{h} + \bar{h}^{-1}\partial_- \bar{h}\,,$$

can choose a gauge: $A_+ = (g^{-1}\partial_+ g + g^{-1}A_+ g)_{\mathfrak{h}}$,

$A_- = (-\partial_- g g^{-1} + gA_- g^{-1})_{\mathfrak{h}}$

remains left-right H gauge symmetry: $h = \bar{h}$

"off-shell" symmetry of corresponding gWZW action

G/H gWZW action with potential:

$$
\begin{aligned}
L = \quad & - \quad \frac{1}{2}\mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \text{WZ term} \\
& - \quad \mathrm{Tr}\big(A_+ \partial_- g g^{-1} - A_- g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-\big) \\
& - \quad \mu^2 \mathrm{Tr}(T g^{-1} T g)
\end{aligned}
$$

Pohlmeyer-reduced theory for F/G coset sigma model

(as first proposed by Bakas, Park, Shin 95)

and thus also for strings on $R_t \times F/G$ or $F/G \times S^1_\psi$

integrable potential: relation at the level of Lax pairs

special case of non-abelian Toda theory:

"symmetric space Sine-Gordon model"

(Hollowood, Miramontes et al 96)

Similar reduction for G PCM or $\frac{G\times G}{G}$ coset leads to G/H theory

with $H = [U(1)]^r$= Cartan of G,

"homogeneous Sine-Gordon model", known to be quantum-integrable

generalizes CSG model ($G = S^3 = SO(3)$)

What to do about A_+, A_-: integrate out or gauge-fix

Reduced equation of motion in the "on-shell" gauge $A_\pm = 0$:

On-shell $\partial_- A_+ - \partial_+ A_- + [A_-, A_+] = 0$ so can set $A_\pm = 0$

$$\partial_-(g^{-1}\partial_+ g) - \mu^2 [T, g^{-1} T g] = 0 \,,$$
$$(g^{-1}\partial_+ g)_{\mathfrak{h}} = 0 \,, \qquad (\partial_- g g^{-1})_{\mathfrak{h}} = 0 \,.$$

$$F/G = SO(n+1)/SO(n) = S^n : \; G/H = SO(n)/SO(n-1)$$

$$g = \begin{pmatrix} k_1 & k_2 & \dots & k_n \\ \dots & \dots & \dots & \dots \end{pmatrix} , \qquad \sum_{1=1}^{n} k_l k_l = 1$$

get (in general non-Lagrangian) EOM for k_m

$$\partial_- \frac{\partial_+ k_\ell}{\sqrt{1 - \sum_{m=2}^n k_m k_m}} = -\mu^2 k_\ell \,, \qquad \ell = 2, \dots, n \,.$$

Linearising around the vacuum $g = 1$ (i.e. $k_1 = 1, \; k_\ell = 0$)

$$\partial_+ \partial_- k_\ell + \mu^2 k_\ell + O(k_\ell^2) = 0$$

massive spectrum: non-trivial S-matrix with H global symmetry

$F/G = SO(n+1)/SO(n) = S^n$:
parametrization of g in Euler angles
$g = e^{T_{n-1}\theta_{n-1}} ... e^{T_2\theta_2} e^{2T\varphi} e^{T_2\theta_2} ... e^{T_{n-1}\theta_{n-1}}$
and integrating out $H = SO(n-1)$ gauge field $A_\pm$
leads to reduced theory that generalizes SG and CSG

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + G_{pq}(\varphi,\theta)\partial_+\theta^p\partial_-\theta^q + \frac{\mu^2}{2}\cos 2\varphi$$

no B_{mn} coupling

similar for $F/G = SO(2,n-1)/SO(1,n-1) = AdS_n$ case:
$G/H = SO(1,n-1)/SO(n-1)$

Bosonic strings on $AdS_n \times S^n$

straightforward generalization:

Lagrangian and the Virasoro constraints

$L = \mathrm{Tr}(P_+^A P_-^A) - \mathrm{Tr}(P_+^S P_-^S)\,,$

$\mathrm{Tr}(P_\pm^S P_\pm^S) - \mathrm{Tr}(P_\pm^A P_\pm^A) = 0$

fix conformal symmetry by

$\mathrm{Tr}(P_\pm^S P_\pm^S) = \mathrm{Tr}(P_\pm^A P_\pm^A) = -\mu^2$

then PR applies independently in each sector:

get direct sum of reduced systems for S^n and AdS_n

linked by Virasoro, i.e. common μ

e.g. for $F/G = AdS_2 \times S^2$:

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$

$AdS_5 \times S^5$ superstring sigma-model

$AdS_5 \times S^5 = \frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)}$

supercoset GS sigma model (Metsaev, AT 98)

$\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$

basic superalgebra $\widehat{\mathfrak{f}} = psu(2,2|4)$

bosonic part $\mathfrak{f} = su(2,2) \oplus su(4) \cong so(2,4) \oplus so(6)$

admits Z_4-grading: (Berkovits, Bershadsky, et al 89)

$$\widehat{\mathfrak{f}} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3 \,, \qquad\qquad [\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{i+j \bmod 4}$$

$\mathfrak{f}_0 = \mathfrak{g} = sp(2,2) \oplus sp(4)$

current ($J = f^{-1}\partial_a f,\ f \in \widehat{F}$) decomposes as

$$J_a = f^{-1}\partial_a f = \mathcal{A}_a + Q_{1a} + P_a + Q_{2a}$$

$$\mathcal{A} \in \mathfrak{f}_0, \quad Q_1 \in \mathfrak{f}_1, \quad P \in \mathfrak{f}_2, \quad Q_2 \in \mathfrak{f}_3 \,.$$

GS Lagrangian:

$$L_{\rm GS} = \frac{1}{2}\,{\rm STr}(\sqrt{-g}g^{ab}P_a P_b + \varepsilon^{ab}Q_{1a}Q_{2b})\,,$$

very simple structure – but not standard coset model:

fermionic currents in WZ term only

leads to κ-symmetry: $\delta_\kappa J_a = \partial_a \epsilon + [J_a, \epsilon]$, $(\delta_\kappa \sqrt{-g}g^{ab})^{ab} = ...$

$\epsilon = \{P_{(+)a}, ik^a_{1(-)}\} + \{P_{(-)a}, ik^a_{2(+)}\}$

conformal gauge:

$$L_{\rm GS} = {\rm STr}[P_+ P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})]$$

$${\rm STr}(P_+P_+) = 0\,, \qquad {\rm STr}(P_-P_-) = 0$$

In terms of current $J = \mathcal{A} + P + Q_1 + Q_2$

$$\text{EOM}: \quad \begin{aligned} &\partial_+ P_- + [\mathcal{A}_+, P_-] + [Q_{2+}, Q_{2-}] = 0\,, \\ &\partial_- P_+ + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] = 0\,, \\ &[P_+, Q_{1-}] = 0\,, \qquad [P_-, Q_{2+}] = 0\,. \end{aligned}$$

Virasoro : $\mathrm{STr}(P_+P_+) = 0\,, \qquad \mathrm{STr}(P_-P_-) = 0$

MC : $\partial_- J_+ - \partial_+ J_- + [J_-, J_+] = 0\,.$

PR procedure: solve first EOM and Virasoro

κ-gauge condition: $Q_{1-} = 0\,, \qquad Q_{2+} = 0$

solves the last (fermionic) pair of EOM

remaining EOM:

$$\partial_+ P_- + [\mathcal{A}_+, P_-] = 0\,, \qquad \partial_- P_+ + [\mathcal{A}_-, P_+] = 0$$

Maurer-Cartan:

$$\partial_+ \mathcal{A}_- - \partial_- \mathcal{A}_+ + [\mathcal{A}_+, \mathcal{A}_-] + [P_+, P_-] + [Q_{1+}, Q_{2-}] = 0\,,$$
$$\partial_- Q_{1+} + [\mathcal{A}_-, Q_{1+}] - [P_+, Q_{2-}] = 0\,,$$
$$\partial_+ Q_{2-} + [\mathcal{A}_+, Q_{2-}] - [P_-, Q_{1+}] = 0\,.$$

as in the bosonic F/G case can fix the "reduction gauge"

$$P_+ = \mu\, T\,, \qquad T = \frac{i}{2}\mathrm{diag}(1, 1, -1, -1|1, 1, -1, -1)$$

$$P_- = \mu\, g^{-1} T g\,, \quad \mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1} A_+ g\,, \quad \mathcal{A}_- = A_-$$

T defines $\mathfrak{h}$ by $[\mathfrak{h}, T] = 0$:

$\mathfrak{h} = su(2) \oplus su(2) \oplus su(2) \oplus su(2)$

new parametrisation: $G = Sp(2,2) \times Sp(4)$-valued field g

and $\mathfrak{h}$-valued field $A_\pm$

MC eqs. become:

$$\begin{aligned}
\partial_-(g^{-1}\partial_+ g + g^{-1}A_+ g) - \partial_+ A_- + [A_-, g^{-1}\partial_+ g + g^{-1} A_+ g]& \\
= -\mu^2 [g^{-1} T g, T] + [Q_{1+}, Q_{2-}]&\,, \\
\partial_- Q_{1+} + [A_-, Q_{1+}] =& \mu [T, Q_{2-}]\,, \\
\partial_+ Q_{2-} + [g^{-1}\partial_+ g + g^{-1} A_+ g, Q_{2-}] =& \mu [g^{-1} T g, Q_{1+}]
\end{aligned}$$

AdS_5 and S^5 sectors now coupled by fermions

remains residual κ-symmetry to be fixed

use T to generalise decomposition of bosonic part

$\mathfrak{f} = T \oplus \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{m}$ to superalgebra $psu(2,2|4)$

$$\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}^{\parallel} \oplus \widehat{\mathfrak{f}}^{\perp}\,, \qquad [T, [T, \widehat{\mathfrak{f}}^{\perp}]] = 0$$

define

$$\Psi_1 = Q_{1+}\,, \qquad \Psi_2 = gQ_{2-}g^{-1}$$

$\Psi_1^\perp, \Psi_2^\perp$ can be set =0 by residual κ-symmetry

remaining fermionic components

$$\Psi_{_R} = \frac{1}{\sqrt{\mu}}\Psi_1^{\parallel}\,, \qquad \Psi_{_L} = \frac{1}{\sqrt{\mu}}\Psi_2^{\parallel}\,,$$

transform under $H \times H$ as $\Psi_{_R} \to \bar{h}^{-1}\Psi_{_R}\bar{h}\,,\ \Psi_{_L} \to h^{-1}\Psi_{_L}h\,.$

equations of motion of reduced theory are thus:

$$\partial_-(g^{-1}\partial_+ g + g^{-1}A_+ g) - \partial_+ A_- + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g]$$
$$= \ -\mu^2[g^{-1}Tg, T] - \mu[g^{-1}\Psi_{_L} g, \Psi_{_R}]\,,$$

$$[T, D_-\Psi_{_R}] = -\mu(g^{-1}\Psi_{_L} g)^{\parallel}\,, \qquad [T, D_+\Psi_{_L}] = -\mu(g\Psi_{_R} g^{-1})^{\parallel}\,.$$

Lagrangian of PR theory for $AdS_5 \times S^5$ superstring

(Grigoriev, AT 07; related work: Mikhailov, Schafer-Nameki 07)

fermionic generalization of "gWZW+ potential" theory for

$$\frac{G}{H} = \frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}$$

$$\begin{aligned} L &= L_{\rm gWZW}(g, A_+, A_-) + \mu^2\, {\rm STr}(g^{-1}TgT) \\ &+ {\rm STr}\left(\Psi_{_L}[T, D_+\Psi_{_L}] + \Psi_{_R}[T, D_-\Psi_{_R}]\right) \\ &+ \mu\, {\rm STr}\left(g^{-1}\Psi_{_L} g \Psi_{_R}\right) \end{aligned}$$

direct sum of PR theories for AdS_5 and S^5

"gluided together" by components of fermions

$$\begin{aligned} L &= \widetilde{L}_{S^5}(g, A_+, A_-) + \widetilde{L}_{AdS_5}(g, A_+, A_-) \\ &+ \psi_{_L} D_+ \psi_{_L} + \psi_{_R} D_+ \psi_{_R} + \mu\ (\text{interaction terms}) \end{aligned}$$

all gauge symmetries fixed; standard kin. terms (cf. GS action)

Comments:

- gWZW model coupled to the fermions interacting minimally and through the "Yukawa term"
- 8 real bosonic and 16 real fermionic independent variables
- 2d Lorentz invariant with $\Psi_{_R}, \Psi_{_L}$ as 2d Majorana spinors
- 2d supersymmetry? yes, at the linearised level, and yes in $AdS_2 \times S^2$ case: $n = 2$ super sine-Gordon
- μ-dependent interaction terms are equal to original GS Lagrangian; gWZW produces MC eq.: path integral derivaion via change from fields to currents?
- quadratic in fermions (like susy version of gWZW); integrating out $A_\pm$ gives quartic fermionic terms (reflecting curvature)
- linearisation of EOM in the gauge $A_\pm = 0$ around $g = \mathbf{1}$ describes 8+8 massive bosonic and fermionic d.o.f. with mass μ: same as in BMN limit
- symmetry of resulting relativistic S-matrix: $H = [SU(2)]^4$ – same as bosonic part of magnon S-matrix symmetry $[PSU(2|2)]^2$

Example: superstring on $AdS_2 \times S^2$

Explicit parametrisation:

$$T = \frac{1}{2} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} .$$

$$g = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & \cos\varphi & i\sin\varphi \\ 0 & 0 & i\sin\varphi & \cos\varphi \end{pmatrix}$$

$$\Psi_{_R} = \begin{pmatrix} 0 & 0 & 0 & i\gamma \\ 0 & 0 & -\beta & 0 \\ 0 & i\beta & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix} , \quad \Psi_{_L} = \begin{pmatrix} 0 & 0 & 0 & \rho \\ 0 & 0 & -i\nu & 0 \\ 0 & \nu & 0 & 0 \\ i\rho & 0 & 0 & 0 \end{pmatrix}$$

PR Lagrangian: same as $n = 2$ supersymmetric sine-Gordon!

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$
$$+ \beta\partial_-\beta + \gamma\partial_-\gamma + \nu\partial_+\nu + \rho\partial_+\rho$$
$$- 2\mu\left[\cosh\phi \ \cos\varphi\,(\beta\nu + \gamma\rho) + \sinh\phi \ \sin\varphi\,(\beta\rho - \gamma\nu)\right] .$$

indeed, equivalent to

$$\widetilde{L} = \partial_+\Phi\partial_-\Phi^* - |W'(\Phi)|^2$$
$$+\psi^*_{L}\partial_+\psi_{L} + \psi^*_{R}\partial_-\psi_{R} + \left[W''(\Phi)\psi_{L}\psi_{R} + W^{*\prime\prime}(\Phi^*)\psi^*_{L}\psi^*_{R}\right] .$$

bosonic part is of $AdS_2 \times S^2$ bosonic reduced model if

$$W(\Phi) = \mu\cos\Phi\,, \qquad |W'(\Phi)|^2 = \frac{\mu^2}{2}(\cosh 2\phi - \cos 2\varphi)\,.$$

$$\psi_{L} = \nu + i\rho\,, \qquad \psi_{R} = -\beta + i\gamma\,,$$

Open questions

- is there 2d susy in $AdS_5 \times S^5$ case? non-trivial question in the presence of potential
- is there UV finiteness? due to potential at least $n = 1$ susy may be needed for it
- better understanding the relationship between the original and the reduced system: symmetries, vacua, values of conserved charges, etc.; which observables can be related?
- integrating out $A_\pm$ fields: related to identification of right vacua at the Lagrangian level; choose asymmetrically gauged form of the gWZW model with fermions?
- Path integral relation to original model?
- Classical S-matrix? Quantum S-matrix? Relation to magnon S-matrix in BA?