# Dressing Giant Strings

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# Outline

1. AdS/CFT: towards a solution of $\mathcal{N} = 4$ super-Yang-Mills
2. Giant Strings
3. The Dressing Method
4. Examples and Applications

# Introduction: Yang-Mills Theory

Of course there are many reasons to be interesting in Yang-Mills theory, including of course its exciting connection to quantum gravity.

The journey towards an analytic solution of this important and rich theory has been long and profitable.

Like in many areas of physics, if we can't solve the theory we're most interested in, we look for a simpler, similar model that we can solve!

This leads us first to consider the $\mathcal{N} = 4$ supersymmetric version of the theory, which has even richer mathematical structure.

Additional simplification occurs in the planar limit $N \to \infty$.

# Roadmap to the Lamppost

| | | |
|---|---|---|
| QCD | | |
| $\downarrow$ | | |
| $\mathcal{N}=4$ | $=$ | strings in $AdS_5 \times S^5$ |
| $\downarrow$ | | |
| planar limit $N \to \infty$ | $=$ | classical strings |

What does it mean to ‘solve’ planar $\mathcal{N} = 4$ theory?

## **solve** /‘sälv/*verb*

At a minimum, to solve planar $\mathcal{N} = 4$ Yang-Mills means that we would like to be able to determine all correlation functions of local gauge invariant operators.

In the planar limit it is sufficient to restrict our attention to single-trace operators. A general gauge-invariant operator is then a trace of the elementary fields and their covariant derivatives

$$\mathcal{O} = \mathrm{Tr}[F^{\mu\nu}\, D_\mu\chi\, X\, \overline{Z}\, F_{\nu\rho}\, D^\sigma\, Y\, D_\sigma\overline{\chi}\, D^\rho Z]$$

and correlation functions are, in general, complicated functions of the ’t Hooft parameter $\lambda$ and the positions of the operators

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\cdots\rangle = F_{ij\cdots}(\lambda; x_1, x_2, \cdots)$$

This problem is still too hard in general, so we start by looking just at two-point functions.

# The Dilatation Operator

Conformal invariance requires any two-point function take the form

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{c}{|x-y|^{2\Delta}}$$

where the number $\Delta$ depends on the operator $\mathcal{O}$ under consideration and is, in general, a complicated function of $\lambda$.

The collection of numbers $\Delta(\lambda)$ appearing in $\mathcal{N}=4$ are the eigenvalues of the dilatation operator, which generates scale transformations on $\mathbb{R}^4$.

The dilatation operator is identified, via radial quantization, with the Hamiltonian of the theory on $\mathbb{R}\times S^3$.

Determining all $\Delta(\lambda)$ $\Leftrightarrow$ Finding the spectrum of the theory on $S^3$

## Integrability

The current 'paradigm' is that all $\Delta(\lambda)$ should be determinable by integrability; a technology familiar from the study of spin chains.

It is instructive to think of a single-trace operator as a spin chain configuration [Minahan & Zarembo]

$$\mathcal{O} = \text{Tr}[F^{\mu\nu}\, D_\mu\chi\, X\, \overline{Z}\, F_{\nu\rho}\, D^\sigma\, Y\, D_\sigma\overline{\chi}\, D^\rho Z]$$

where the different fields (and their derivatives) are the different 'directions' in which each 'spin vector' can point.

Under radial evolution (generated by the dilatation operator), the spin vectors on different sites can 'interact' with each other.

Planar $L$-loop Feynman diagrams give rise to interactions between spins up to $L$ sites apart from each other.

# The S-matrix

An important property of integrable theories is that they can be solved (i.e., the eigenvalues of the Hamiltonian can be found) once we know the 2-particle S-matrix.

One way to set up this scattering problem is to choose a 'ferromagnetic' ground state

$$\cdots ZZZZZZZZZ \cdots \quad = \quad Z^J$$

This corresponds to a BPS operator whose dimension $\Delta = J$ is protected from quantum corrections. Furthermore we take the infinite volume limit $J \to \infty$ so that we can set up initial and final states of the elementary excitations ('magnons') of the spin chain.

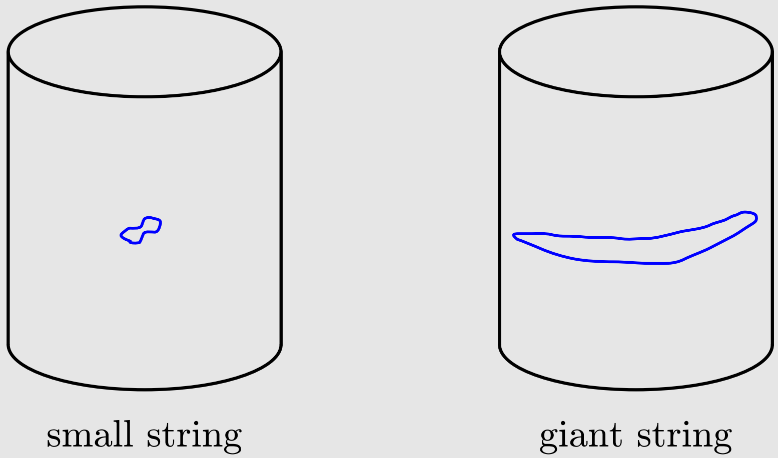
[Berenstein, Maldacena, Nastase; Staudacher, Beisert; Hofman, Maldacena, and many others...]

## Roadmap to the Lamppost

| | | |
|---|---|---|
| QCD | | |
| $\downarrow$ | | |
| $\mathcal{N}=4$ | $=$ | strings in $AdS_5 \times S^5$ |
| $\downarrow$ | | $\downarrow$ |
| planar limit $N \to \infty$ | $=$ | classical string theory |
| $\downarrow$ | | $\downarrow$ |
| two point functions $($spectrum on $S^3)$ | $=$ | string spectrum (in global time) |
| $\downarrow$ | | $\downarrow$ |
| "infinitely long" operators | $=$ | "giant" strings |

# Giant Strings

Short operators are dual to small quantum fluctuations of $AdS_5 \times S^5$, while long operators are dual to ‘giant strings’.



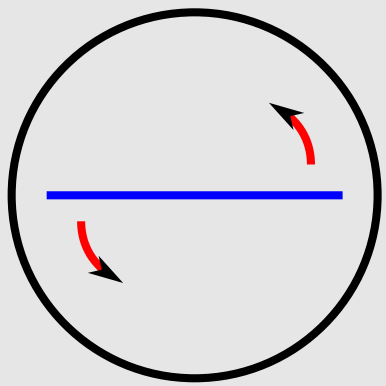

A giant string is one which is comparable in size to anti-de Sitter space. In the $\lambda \to \infty$ limit these are highly excited string states which can be treated classically in the string sigma model on $AdS_5 \times S^5$.

These states necessarily have large energy (as well as other global charges, such as angular momentum), scaling like $\sqrt{\lambda}$.

# Example: The GKP spinning string

The prototypical giant string is the GKP spinning string, which has long played an important role in quantiative checks of AdS/CFT.

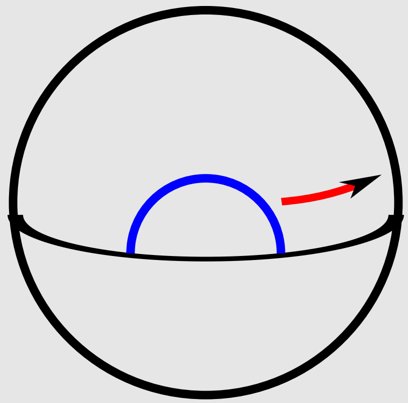
[Gubser, Klebanov, Polyakov]



(top view of $AdS_5$)

# Example: The Giant Magnon

Another, more recent, important example is the giant magnon. This is a string which sits at a point inside $AdS$ and is stretched along the $S^5$. [Hofman, Maldacena]



(side view of $S^5$)

The two 'endpoints' of the string move around the equator of $S^5$ at the speed of light. (Attach several of these end-to-end to make a closed string.)

It is dual to a single elementary magnon excitation of the spin chain in gauge theory, with momentum $p$ given by the angular separation between the two endpoints.

# Dressing Method

Consider the principal chiral model, a matrix field $g(z, \bar{z})$ subject to the equation of motion

$$\bar{\partial}(\partial g\, g^{-1}) + \partial(\bar{\partial} g\, g^{-1}) = 0$$

Given *any* solution $g$ the goal is to construct a new solution $g' = \chi g$ by choosing $\chi$ approrpiately.

In fact such a $\chi$ can be constructed systematically as follows. [Zakharov, Mikhailov].

The first step is to recast the above second-order equation into a linear first-order system for a new field $\Psi$ at the cost of introducing a new auxiliary variable $\lambda$:

$$i\bar{\partial}\Psi = \frac{A\Psi}{1+\lambda}, \qquad i\partial\Psi = \frac{B\Psi}{1-\lambda}$$

# Dressing Method

$$\bar{\partial}(\partial g\, g^{-1}) + \partial(\bar{\partial} g\, g^{-1}) = 0$$

$$i\bar{\partial}\Psi = \frac{A\Psi}{1+\lambda}, \qquad i\partial\Psi = \frac{B\Psi}{1-\lambda}$$

$\Longrightarrow$ Given any solution $g$ we can take

$$A = i\bar{\partial} g\, g^{-1}, \qquad B = i\partial g\, g^{-1}$$

and solve for $\Psi(\lambda)$ with initial condition $\Psi(0) = g$.

$\Longleftarrow$ Given any collection $(\Psi(\lambda), A, B)$ which satisfies the auxiliary system, it is easy to check that $\Psi(0)$ is guaranteed to satisfy the principal chiral equations of motion.

# Dressing Method

The auxiliary system can be used to show that the ‘gauge transformation’

$$\Psi(\lambda) \to \chi(\lambda)\Psi(\lambda), \qquad \chi(\lambda) = 1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P$$

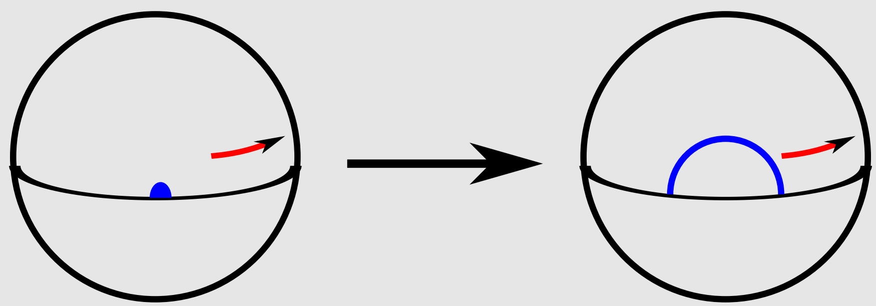
maps solutions into solutions, where $\lambda_1$ is an arbitrary complex parameter and $P$ is a projection operator onto the subspace spanned by $\Psi(\bar{\lambda}_1)e_1$ for any constant vector $e_1$.

Summary: given any initial solution (which we sometimes call the ‘vacuum’), one can systematically construct (by successive applications) more complicated solutions labelled by the parameters $(\lambda_i, e_i)$.

Successful application of this technique relies on the idenficiation of a suitable ‘vacuum’ solution to use as a seed for the dressing.

# Dressing the Giant Magnon

For the giant magnon it is not difficult to guess the appropriave ‘vacuum’: the giant magnon is an excitation of the solution



a pointlike string moving around the equator of the sphere at the speed of light, which is dual to the gauge theory operator $\mathrm{Tr}[Z^J]$.

In this particular case, there is another way to see what is going on: string theory on $\mathbb{R}_{\mathrm{time}} \times S^2$ is classically equivalent (via Pohlmeyer reduction) to the sine-Gordon model.

## Sine-Gordon

$$\text{Pohlmeyer map} \qquad \partial X \cdot \bar{\partial} X = \cos\alpha$$

Given a solution $X^i$ of the string sigma-model it is simple to calculate the corresponding sine-Gordon solution $\alpha$ under this map. The giant magnon is equivalent to the sine-Gordon soliton.

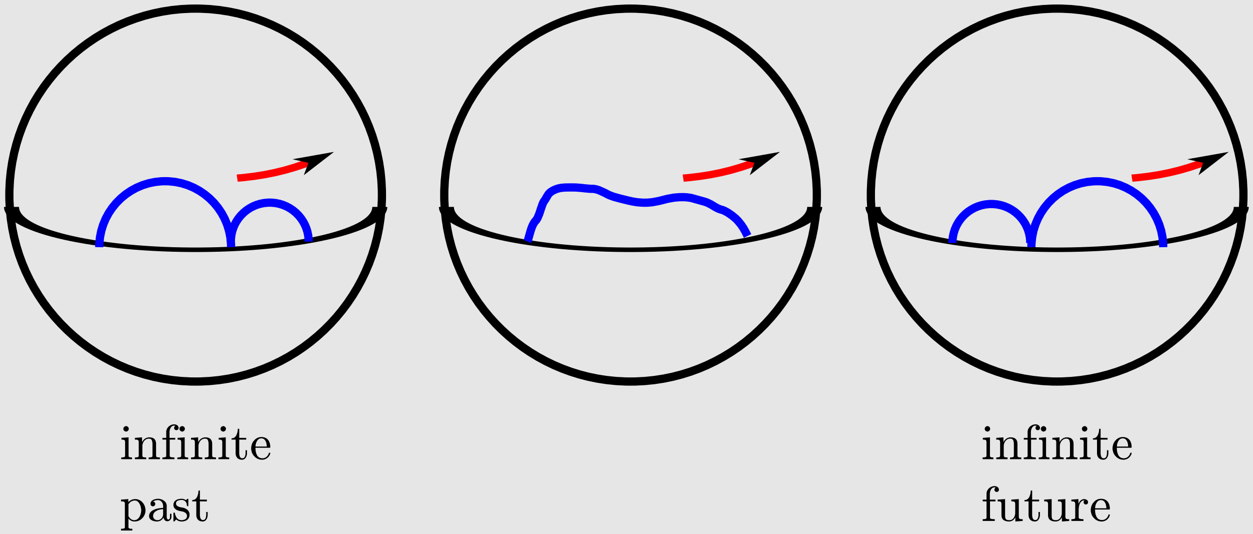
Of course in the sine-Gordon model an explicit formula for the $k$-soliton solutions is known.

However to invert the map—to find the string solution from a known sine-Gordon solution—requires solving a non-linear differential equation which in general is impossible to carry out directly.

[Chen, Dorey and Okamura was able to find the solution describing BPS bound states of giant magnons this way because the variables separate.]

# Simple Example: Magnon Scattering

Using the dressing method it straightforward to write down a 2-magnon scattering solution (or more, if desired) [0607009].




It is an interesting open problem to dtermine whether one can write down an explicit analytic expression for a $k$-magnon scattering solution.

## Applications

Having explicit solutions for these kinds of complicated giant string solutions is useful for several reasons.

1) They allow the calculation of quantum corrections to the magnon scattering phase [0704.2389], confirming the Hernandez-Lopez phase factor at 1-loop [Heng-Yu Chen, Nick Dorey, Rui F. Lima Matos]. (The equivalence with sine-Gordon theory is at the classical level only.)

2) Even at the classical level, they allow useful calculation of scattering phase shifts for cases where the dual gauge theory description is unknown or poorly understood [0611033, 0710.2300].

3) The explicit solution [0607009, 0611033] for the scattering of two BPS magnon boundstates exhibits a symmetry property that helps to elucidate the analytic structure of the magnon S-matrix (there are no non-BPS magnon boundstates) [Dorey, Hofman, Maldacena].

# Other Dressings I

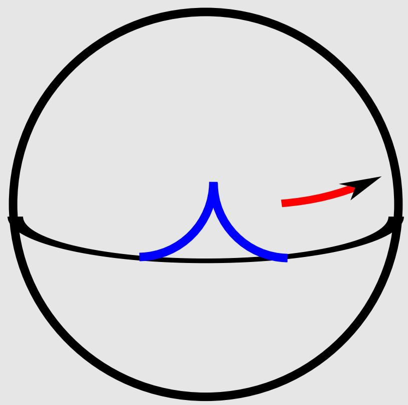
The giant magnon itself carries angular momentum $J_1$ on the sphere. A BPS bound state of $J_2$ giant magnons carries an additional angular momentum $J_2$ on some axis of $S^5$ [Dorey] .

A solution describing a superposition of two such states with momentum $p = \pi$ and carrying orthogonal angular momenta on the $S^5$ was found 'the hard way' [Kruczenski, Russo, Tseytlin].

By applying the dressing method we can write down explicit formulas for completely general superpositions, describing the scattering of magnon bound states carrying arbitrary momenta and with arbitrary orientations of the $J_2$ angular momentum axes [0611033].

# Other Dressings II

Another type of giant string are ‘giant spikes’ [Ishizeki, Kruczenski]



(side view of $S^5$)

This solution can be obtained by dressing the ‘vacuum’ that consists of a static string infinitely wound around the equator of the sphere.

## Scattering of Giant Spikes

The spin chain description of these giant spikes is not completely clear but it has been suggested that they should be thought of as excitations of the antiferromagnetic state $\mathrm{Tr}[(Z\bar{Z})^J]$[Zarembo; Roiban, Tirziu, Tseytlin].

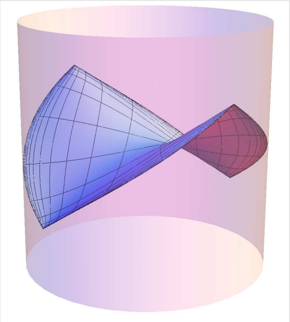
Further application of the dressing method allows for the construction of explicit solutions describing the scattering of spikes, which allow their classical scattering phase shift to be computed.

Suprisingly, the result comes out to be *precisely* identical (as a function of the dressing parameter $\lambda_1$ as the result for the giant magnon (even though intermediate steps in the calculation are quite different).

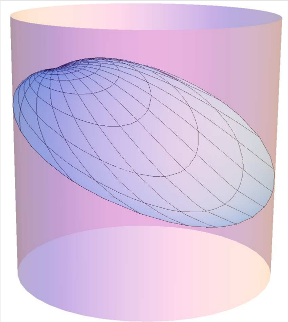
# Other Dressings III

So far I have discussed strings whose induced worldsheet metric is Lorentzian and are dual to energy eigenstates in the dual gauge theory.

The dressing method is also applicable to the problem of finding minimal area Euclidean worldsheets in $AdS$, which are dual to Wilson loops.



giant gluon



a dressed giant gluon

As an example one can dress the 'giant gluon' solution that was recently used by Alday and Maldacena to study gluon scattering amplitudes at strong coupling. The result is a worldsheet which traces out a timelike curve on the boundary of $AdS$.

# On the Choice of Vacuum

One slightly unsatisfying aspect of these particular solutions, that makes it difficult to interpret their purpose in life, is that it is not clear what the ‘vacuum’ for this class of solutions should be.

An equivalent question is whether the giant gluon solution (originally studied by Kruczenski in a related context) itself can be understood as a ‘dressed’ version of some more elementary solution.

A similar open question holds for the GKP spinning string...

## Summary

Classical string solutions of many various types have long played an important role in exploring the structure of the AdS/CFT correspondence.

A large number of important classical string solutions from the literature ([Hofman, Maldacena; Chen, Dorey, Okamura; Ishizeki, Kruczenski; etc.]) can be obtained by 'dressing' a suitable vacuum solution.

Moreover, the dressing method allows one to systematically exploit the classical integrability of string theory on $AdS_5 \times S^5$ to construct explicit analytic solutions that would be impossible to obtain by solving the relevant non-linear string equations of motion 'by hand.'