Quantization of the open string on exact plane waves and non-commutative wave fronts

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Motivation

- **On-going issue** in the NC community: Does non-commutativity play a rôle in connection with gravity?

  ... The **hope** (=? expectation) is the answer to this question to be yes! ...

  It seems reasonable to look for an answer within string theory.

  One step along this direction is to invoke the **Penrose limit and look for non-commutative spaces in *pp* strings**.

- **Genuine interest in *pp* strings** within string theory.

Problem

- In particular, I will consider the ***pp* background**

$$ds^2 = dx^+ dx^- + m^2 \left[(x^1)^2 - (x^2)^2 \right] (dx^+)^2 + (dx^i)^2 + (dx^a)^2$$

$$i = 1, 2 \qquad a = 3, \ldots D-1$$

$$B_{12} = B, \quad \text{any other } B_{\mu\nu} = 0 \qquad \Phi = \text{const.}$$

and quantize the open string on it.

This is a **Non-singular exact solution to all orders in α'**.

- The results to be presented trivially extend to e.g. 10 dimensions

$$ds^2 = dx^+ dx^- + \sum_{k=1}^{4} m_k^2 \left[(x^k)^2 - (x^{k+4})^2 \right] (dx^+)^2 + \sum_{k=1}^{8} (dx^k)^2$$

$$B_{15}, B_{26}, B_{37}, B_{48} \neq 0 \,.$$

Compare with maximally supersymmetric *pp* IIB superstring.

$$ds^2 = dx^+ dx^- + m^2 \left[\sum_{k=1}^{4} (x^k)^2 + \sum_{k=5}^{8} (x^k)^2 \right] (dx^+)^2 + \sum_{k=1}^{8} (dx^i)^2$$

$$F_5 = m \, dx^+ \wedge \left(dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8 \right) .$$

Strategy: use canonical quantization in lightcone and conformal gauge:

1. **Solution to the equations of motion for the classical string.**
2. **Canonical quantization**: by inverting the symplectic form.
3. **The Hamiltonian spectrum and the Fock-Hilbert space**

Let us anticipate some results:

4. **Commutation relations** (in light-cone gauge $x^+ = \kappa\tau$ – wave fronts)

$$\big[X^i(\tau,\sigma), P_j(\tau,\sigma')\big] = i\alpha'\,\delta^i{}_j\,\delta(\sigma-\sigma')$$

$$\big[X^1(\tau,\sigma), X^2(\tau,\sigma')\big] = i\,\Theta(\sigma,\sigma')$$

– **$\Theta(\sigma,\sigma') \neq 0$ for all σ,σ' (i.e. all along the string, not only at endpoints).**
  $\sim$ Non-commutativity without/outside branes.

– **As $m \to 0$, Minkowski spacetime results are recovered. Non-commutativity is confined back to the string endpoints.**

1. Classical string.

$$S = \frac{1}{4\pi\alpha'} \int d\tau\, d\sigma \left(\sqrt{-\gamma}\, \gamma^{rs}\, G_{\mu\nu}\, \partial_r X^\mu\, \partial_s X^\nu + \varepsilon^{rs} B_{\mu\nu}\, \partial_r X^\mu\, \partial_s X^\nu + \alpha' \sqrt{-\gamma}\, R\, \Phi \right).$$

In lightcone and conformal gauge: $\boldsymbol{X^+ = \kappa\tau}$ and (from the action)

Equations of motion

$$\Box X^- + 4m^2\kappa \left(X^1 \partial_\tau X^1 - X^2 \partial_\tau X^2 \right) = 0$$

$$\Box X^1 + m^2\kappa^2 X^1 = 0$$

$$\Box X^2 - m^2\kappa^2 X^2 = 0$$

$$\Box X^a = 0$$

Boundary conditions

$$\partial_\sigma X^- \big|_{\sigma=0,\pi} = 0$$

$$\partial_\sigma X^1 - B\, \partial_\tau X^2 \big|_{\sigma=0,\pi} = 0$$

$$\partial_\sigma X^2 + B\, \partial_\tau X^1 \big|_{\sigma=0,\pi} = 0$$

$$\partial_\sigma X^a \big|_{\sigma=0,\pi} = 0$$

Virasoro constraints

$$\kappa\, \partial_\tau X^- = m^2\kappa^2 \left[\left(X^1\right)^2 - \left(X^2\right)^2 \right] + \left(\partial_\tau X^i\right)^2 + \left(\partial_\tau X^a\right)^2 + \left[\left(\partial_\sigma X^i\right)^2 + \left(\partial_\sigma X^a\right)^2 \right]$$

$$\kappa\, \partial_\sigma X^- = 2 \left(\partial_\tau X^i\right)\left(\partial_\sigma X^i\right) + 2 \left(\partial_\tau X^a\right)\left(\partial_\sigma X^a\right)$$

- a-directions: same as in Minkowski spacetime. Solution is well-known.
- Equations for X^1, X^2 and Virasoro constraints $\Rightarrow$ Equation for X^+.

- The **solution** for the boundary problem for X^1, X^2 is

$$\boldsymbol{X^1(\tau,\sigma)} = i \sum_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}_\pm} \boldsymbol{c_\lambda} \frac{\alpha}{\lambda B} \left(\cos\beta\sigma + \frac{\sin\beta\pi}{\cos\beta\pi \mp 1} \sin\beta\sigma \right) e^{-i\lambda\tau}$$

$$\boldsymbol{X^2(\tau,\sigma)} = - \sum_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}_\pm} \boldsymbol{c_\lambda} \left(\frac{\cos\alpha\pi \pm 1}{\sin\alpha\pi} \cos\alpha\sigma + \sin\alpha\sigma \right) e^{-i\lambda\tau}$$

$$\boldsymbol{\Lambda_+} = \{\text{solutions of } \boldsymbol{F_+(\lambda) = 0}, \ \sin\alpha\pi \sin\beta\pi \neq 0\}$$

$$\boldsymbol{\Lambda_-} = \{\text{solutions of } \boldsymbol{F_-(\lambda) = 0}, \ \sin\alpha\pi \sin\beta\pi \neq 0\}$$

$$\boldsymbol{F_\pm(\lambda)} = \frac{\alpha\beta}{\lambda^2 B^2} - \frac{(\cos\alpha\pi \pm 1)(\cos\beta\pi \mp 1)}{\sin\alpha\pi \ \sin\beta\pi}$$

$$\boldsymbol{\alpha} = \sqrt{\lambda^2 - m^2\kappa^2}$$

$$\boldsymbol{\beta} = \sqrt{\lambda^2 + m^2\kappa^2}$$

$\boldsymbol{c_\lambda}$ = arbitrary constants of integration

$\boldsymbol{\Lambda_+}$ and $\boldsymbol{\Lambda_-}$ are disjoint: $\boldsymbol{F_+(\lambda)}$ and $\boldsymbol{F_-(\lambda)}$ do not have common roots.

The eigenvalue equations $\boldsymbol{F_\pm(\lambda) = 0}$ have infinitely many solutions:

- Come in pairs $(\lambda, -\lambda)$ since $\boldsymbol{F_\pm}$ are functions of $\boldsymbol{\lambda^2}$.
- $\boldsymbol{\lambda^2 > 0} \Rightarrow$ Real $\lambda \Rightarrow e^{-i\lambda\tau} =$ oscillations in τ

  Finite number (≥ 1, depends on the value of $m\kappa$) of them with $|\lambda| < m\kappa$. Infinitely many with $|\lambda| > m\kappa$. In fact, for $|\lambda| \gg m\kappa$, there are 2 solutions in the neighborhood of every large enough integer n

$$\lambda_n^{(1,2)} = n \left[1 \pm \frac{m^2\kappa^2}{2\,n^2} \, \frac{1 - B^2}{1 + B^2} + \mathcal{O}\left(\frac{m^4\kappa^4}{n^4} \right) \right].$$

- $\boldsymbol{\lambda^2 < 0} \Rightarrow$ Imaginary $\lambda \Rightarrow e^{-i\lambda\tau} =$ non-oscillatory τ-dependence

  Finite number (≥ 1, depends on the value of $m\kappa$) of them, all with $|\lambda| < m\kappa$.

Conclusion: infinitely many oscillations (real λ) and a finite number of non-oscillatory position-momentum type degrees of freedom (imaginary λ).

Comment. This solution exists for all values of $m\kappa$.
(i) If $m^2\kappa^2 =$ even integer, there are a few additional oscillation modes.
(ii) If $m\kappa =$ integer, there is in addition one more position-momentum type mode.
To keep formulas simple, I will not write them here.

• Momenta.

$$\boldsymbol{P_i(\tau,\sigma)} = -\frac{\delta S}{\delta\big(\partial_\tau X^i(\tau,\sigma)\big)} = \frac{1}{2\pi\alpha'}\left(\partial_\tau X^i - B\,\epsilon_{ij}\,\partial_\sigma X^j\right) = \sum_{\lambda\in\boldsymbol{\Lambda}_\pm}\cdots$$

From the resulting mode expansions the total momentum $p_i = \int_0^\pi d\sigma P_i$ is

$$\boldsymbol{p_1(\tau)} = \frac{1}{\pi\alpha'}\sum_{\lambda\in\boldsymbol{\Lambda}_-}\frac{B\,c_\lambda}{\beta^2}\,\frac{\cos\alpha\pi - 1}{\sin\alpha\pi}\,e^{-i\lambda\tau}$$

$$\boldsymbol{p_2(\tau)} = \frac{1}{\pi\alpha'}\sum_{\lambda\in\boldsymbol{\Lambda}_+}\frac{c_\lambda}{\lambda\alpha}\,e^{-i\lambda\tau}$$

– $p_i(\tau)$ are not conserved. Expected!

– $p_i(\tau)$ are collective. Receive contributions from all modes.

– Upon quantization, they will not play a significant rôle.

3. Quantization.

$$S = \int d\tau\, d\sigma \left[-\left(P_i\, \partial_\tau X^i + P_a\, \partial_\tau X^a\right) + \mathcal{F} \right]$$

$$\frac{1}{2\pi\alpha'} \left[\left(2\pi\alpha' P_i + B\, \varepsilon_{ij}\, \partial_\sigma X^j\right)^2 + \left(2\pi\alpha' P_a\right)^2 + \left(\partial_\sigma X^i\right)^2 + \left(\partial_\sigma X^a\right)^2 \right]$$

Symplectic form: $\boldsymbol{\Omega} = \int_0^\pi d\sigma \left(\mathbf{d}P_i \wedge \mathbf{d}X^i + \mathbf{d}P_a \wedge \mathbf{d}X^a\right) =: \boldsymbol{\Omega_i} + \Omega_a$

$$\boldsymbol{\Omega_i} = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma\ \mathbf{d}(\partial_\tau X^i) \wedge \mathbf{d}X^i + \frac{B}{2\pi\alpha'}\ \mathbf{d}X^1 \wedge \mathbf{d}X^2 \Big|_{\sigma=0}^{\sigma=\pi}$$

Using the mode expansions and integrating over $d\sigma$,

$$\boldsymbol{\Omega_i} = \frac{1}{2} \sum_{\lambda \in \Lambda_\pm} \boldsymbol{\Omega_{\lambda,-\lambda}}\ \mathbf{d}\boldsymbol{c_\lambda} \wedge \mathbf{d}\boldsymbol{c_{-\lambda}}$$

$$\boldsymbol{\Omega_{\lambda,-\lambda}} = \frac{i}{\pi\alpha'} \underbrace{(-)\, \frac{\lambda\alpha\,(\cos\alpha\pi \pm 1)}{\sin\alpha\pi} \left[\frac{2\,(m\kappa)^4}{\lambda^2\alpha^2\beta^2} \pm \frac{\pi}{\alpha \sin\alpha\pi} \mp \frac{\pi}{\beta \sin\beta\pi} \right]}_{=:\ \boldsymbol{f(\lambda) \neq 0}\ \ \textbf{for all}\ \boldsymbol{\lambda \in \Lambda_\pm}}$$

Ω is non-singular. Its inverse exists

Quantization: Amplitudes c_λ become operators, with

$$c_\lambda^\dagger = c_{-\lambda} \quad \lambda \text{ real}$$

$$c_\lambda^\dagger = c_\lambda \quad \lambda \text{ imaginary}$$

$$\left[\, c_\lambda, c_{\lambda'} \,\right] = i\left(\Omega^{-1}\right)_{\lambda\lambda'} = -\frac{\pi\alpha'}{f(\lambda)}\,\delta_{\lambda+\lambda',0}$$

Comment. Symplectic form has “canonical” form (only $\lambda + \lambda' = 0$ contributes).

4. Hamiltonian and Fock-Hilbert space

We want to solve $H|\psi\rangle = E|\psi\rangle$, where

$$
\begin{aligned}
H &= \boldsymbol{H_i} + H_a \\
\boldsymbol{H_i} &= \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \left\{ \left(\partial_\tau X^i\right)^2 + \left(\partial_\sigma X^i\right)^2 - m^2\kappa^2 \left[(X^1)^2 - (X^2)^2 \right] \right\} \\
H_a &= \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \left[\left(\partial_\tau X^a\right)^2 + \left(\partial_\sigma X^a\right)^2 \right]
\end{aligned}
$$

- $H_a|\psi_a\rangle = E_a|\psi_a\rangle$ Solution is well-known
  - plane waves of momentum p_a
  - harmonic oscillators of integer frequency.
  - E_a has a normal ordering constant $\dfrac{D-4}{24}$.

- $\boldsymbol{H_i|\psi_i\rangle = E_i|\psi_i\rangle}$

$$\boldsymbol{H_i} = \frac{1}{2\pi\alpha'}\sum_{\lambda\in\Lambda_\pm} \lambda\, f(\lambda)\, c_\lambda\, c_{-\lambda}$$

$$= \underbrace{\frac{1}{\pi\alpha'}\sum_{\substack{\lambda\in\Lambda_\pm \\ \mathbf{Re}\,\lambda>0}} \lambda\, f(\lambda) : c_\lambda^\dagger c_\lambda :}_{} - \underbrace{\frac{1}{2}\sum_{\substack{\lambda\in\Lambda_\pm \\ \mathbf{Re}\,\lambda>0}} \lambda}_{} + \underbrace{\sum_{\substack{\lambda\in\Lambda_\pm \\ \mathbf{Im}\,\lambda>0}} \mathrm{sign}\big[\lambda f(\lambda)\big]\,\big(\hat{p}_\lambda^2 - \hat{q}_\lambda^2\big)}_{}$$

Harmonic oscillators of frequency λ

normal ordering

$$c_{\pm\lambda} = \sqrt{\frac{\pi\alpha'}{2\,|\lambda f(\lambda)|}}\,\big(\hat{q}_\lambda \pm \hat{p}_\lambda\big)$$

$\mathrm{Im}\,\lambda > 0$

↓

For every λ ($\mathrm{Im}\lambda > 0$), Hamiltonian for an inverted harmonic oscillator

$$\left.\begin{aligned} |\psi_\mathrm{I}\rangle &= \prod_{\mathrm{Im}\lambda>0} |\psi_\lambda\rangle \\ E_\mathrm{I} &= \sum_{\mathrm{Im}\lambda>0} \mathrm{sign}\big[\lambda f(\lambda)\big]\, E_\lambda \end{aligned}\right\} \qquad \left(\frac{d^2}{dq_\lambda^2} + q_\lambda^2 + E_\lambda\right)\psi_\lambda(q_\lambda) = 0$$

Does not have bound sates. Its solutions are scattering states

$$\psi_\lambda(q_\lambda) = C_1\, e^{-iq_\lambda^2/2}\, q_\lambda\, \Phi\Big(\tfrac{3}{4} + \tfrac{iE}{4}\,, \tfrac{3}{2}\,; iq_\lambda^2\Big) + C_2\, e^{-iq_\lambda^2/2}\, \Phi\Big(\tfrac{1}{4} + \tfrac{iE}{4}\,, \tfrac{1}{2}\,; iq_\lambda^2\Big)$$

$$\searrow \qquad\qquad \swarrow$$

regular at $q_\lambda = 0$

at $q_\lambda \to \infty$ superpositions of $\frac{1}{\sqrt{|q_\lambda|}} \exp\Big[\pm \frac{i}{4}\big(E_\lambda \ln q_\lambda^2 + 2q_\lambda^2\big)\Big]$

$\longrightarrow$ $\psi_\lambda(q_\lambda)\, e^{-iE_\lambda \tau}$ scattering states in 1,2-directions

4. Canonical commutation relations and non-commutativity

- $\left[X^1(\tau,\sigma), X^2(\tau,\sigma')\right] = \frac{1}{2B}\sum_{\lambda\in\Lambda_\pm}\frac{\alpha}{\lambda\, f(\lambda)}\left(\cos\beta\sigma + \frac{\sin\beta\pi}{\cos\beta\pi\mp 1}\,\sin\beta\sigma\right)$

$$\times\left(\frac{\cos\alpha\pi\pm 1}{\sin\alpha\pi}\,\cos\alpha\sigma' + \sin\alpha\sigma'\right) =: i\,\Theta(\sigma,\sigma') \neq 0$$

- **Case $|m\kappa| \ll 1$.** The mode eigenvalues λ can be found as power series in $m\kappa$:

$$\Lambda_- = \{\pm i\lambda^{\mathrm{I}}, \lambda_-^{(n)}\} \qquad \Lambda_+ = \{\pm\lambda^{\mathrm{R}}, \lambda_+^{(n)}\} \qquad n = \pm 1, \pm 2, \ldots$$

$$\lambda^{\mathrm{I}} = \frac{m\kappa}{\sqrt{1+B^2}}\left[1 + \frac{(m\kappa)^2}{12}\,\frac{\pi^2 B^2}{1+B^2} + \frac{(m\kappa)^4}{1440}\,\frac{\pi^4 B^2(5B^2-24)}{(1+B^2)^2} + \mathcal{O}\left(m^6\kappa^6\right)\right]$$

$$\lambda^{\mathrm{R}} = \frac{m\kappa}{\sqrt{1+B^2}}\left[1 - \frac{(m\kappa)^2}{12}\,\frac{\pi^2 B^2}{1+B^2} + \frac{(m\kappa)^4}{1440}\,\frac{\pi^4 B^2(5B^2-24)}{(1+B^2)^2} + \mathcal{O}\left(m^6\kappa^6\right)\right]$$

$$\lambda_\pm^{(n)} = n\left[1 \pm (-)^n\,\frac{m^2\kappa^2}{2\,n^2}\,\frac{1-B^2}{1+B^2} - \frac{m^4\kappa^4}{8\,n^4}\,\frac{B^4-6B^2+1}{(1+B^2)^2} + \mathcal{O}\left(m^6\kappa^6\right)\right].$$

$$\boldsymbol{\Theta(\sigma,\sigma')} = \sum_{k=0}^{\infty} \boldsymbol{\Theta_{2k}(\sigma,\sigma')}\,(m\kappa)^{2k}$$

$\downarrow$

Functions of σ and σ'

Involve powers of σ and σ' and $\displaystyle\sum_{n=1}^{\infty} \frac{\begin{pmatrix}\cos n\sigma \\ \sin n\sigma\end{pmatrix} \times \begin{pmatrix}\cos n\sigma' \\ \sin n\sigma'\end{pmatrix}}{\boldsymbol{n}^{\text{positive integer}}}$

$$\boldsymbol{\Theta_0(0,0) = -\Theta_0(\pi,\pi) = \frac{\alpha'\pi B}{1+B^2}}$$

$$\boldsymbol{\Theta_0(\sigma,\sigma') = 0 \quad \text{for } \sigma+\sigma' \neq 0, 2\pi}$$

As in Minkowski spacetime

[Chu-Ho, Seiberg-Witten]

It comes out of the calculation!

$$\begin{aligned}\boldsymbol{\Theta_2(\sigma,\sigma')} \;=\; & \frac{\alpha' B}{(1+B^2)^2}\left\{B^2\left[-\frac{\sigma}{6}\left(\sigma^2-3\sigma'^2\right)+\frac{\pi}{4}\left(\sigma^2-\sigma'^2-2\sigma\sigma'\right)-\frac{\pi}{12}\left(\sigma-3\sigma'\right)\right]\right.\\ & -\frac{\sigma}{12}\left(7\sigma^2+9\sigma'^2\right)+\frac{\pi}{8}\left(7\sigma^2+3\sigma'^2+6\sigma\sigma'\right)-\frac{\pi^2}{4}\left(3\sigma+\sigma'\right)+\frac{\pi^3}{6}\\ & \left.+\frac{\pi}{8}\left|\sigma-\sigma'\right|\left[2B^2\left(\sigma^2+\sigma'-\pi\right)+5\sigma-\sigma'-2\pi\right]\right\}.\end{aligned}$$

$\boldsymbol{\neq 0}$ **for arbitrary** $\boldsymbol{\sigma,\sigma'} \Rightarrow$ **Non-commutativity** $\displaystyle\frac{\textbf{outside}}{\textbf{without}}$ **branes!**

As $\boldsymbol{m \to 0}$, **NC gets confined to endpoints.**

- Similarly

$$\left[\boldsymbol{X}^i(\boldsymbol{\tau},\boldsymbol{\sigma}),\boldsymbol{P}_j(\boldsymbol{\tau},\boldsymbol{\sigma}')\right] = \sum_{k=0}^{\infty} \boldsymbol{\mathcal{C}}^i{}_{\boldsymbol{j,2k}}(\boldsymbol{\sigma},\boldsymbol{\sigma}')(m\kappa)^{2k}$$

$$\boldsymbol{\mathcal{C}}^i{}_{\boldsymbol{j,0}} = \boldsymbol{i\alpha'\,\delta^i_j\,\delta(\sigma-\sigma')}$$

$$\boldsymbol{\mathcal{C}}^i{}_{\boldsymbol{j,2}} = \boldsymbol{0}$$

$$\boldsymbol{\cdots}$$

- (Total) normal ordering constant is also a power series in $m\kappa$:

$$K = \frac{D-4}{24} + \sum_{\substack{\lambda\in\Lambda_\pm \\ \mathbf{Re}\,\lambda>0}} \lambda$$

$$= \frac{D-2}{24} - \frac{m\kappa}{2\sqrt{1+B^2}}\left[1 - \frac{(m\kappa)^2}{12}\,\frac{\pi^2 B^2}{1+B^2} + \mathcal{O}\left(m^3\kappa^3\right)\right]$$

It looks as if there is room for non-commutativity in connection with gravity within string theory.

- **The low energy-limit (Seiberg-Witten maps).**
- **The Penrose limit.**

Implications for spacetime singularities, etc?