The dilatation operator of $\mathcal{N} = 4$ SYM at 4-loops and beyond

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Outline

- Anomalous dimensions, Integrability and such
- What type of calculations are easy
- Higher loop Hamiltonians
- The calculation of the 4-loop Hamiltonian in the $SU(2)$ sector
- Prospects at higher loops

Anomalous dimensions: govern RG flow of gauge-inv. operators

RG flow of parton distributions

OPE

characterize IR singularities

for CFT: ($\mathcal{N}=4$ SYM) part of the definition of the theory

- spectral problem of $\mathcal{N}=4$ SYM: find all anomalous dimensions

- $\mathcal{O}_n$ "good" operators – diagonal RG flow $\langle \mathcal{O}_n(0)\mathcal{O}_m(x)\rangle = \dfrac{\delta_{mn}}{|x|^{2\Delta_n}}$

  available tricks for special "good" operators

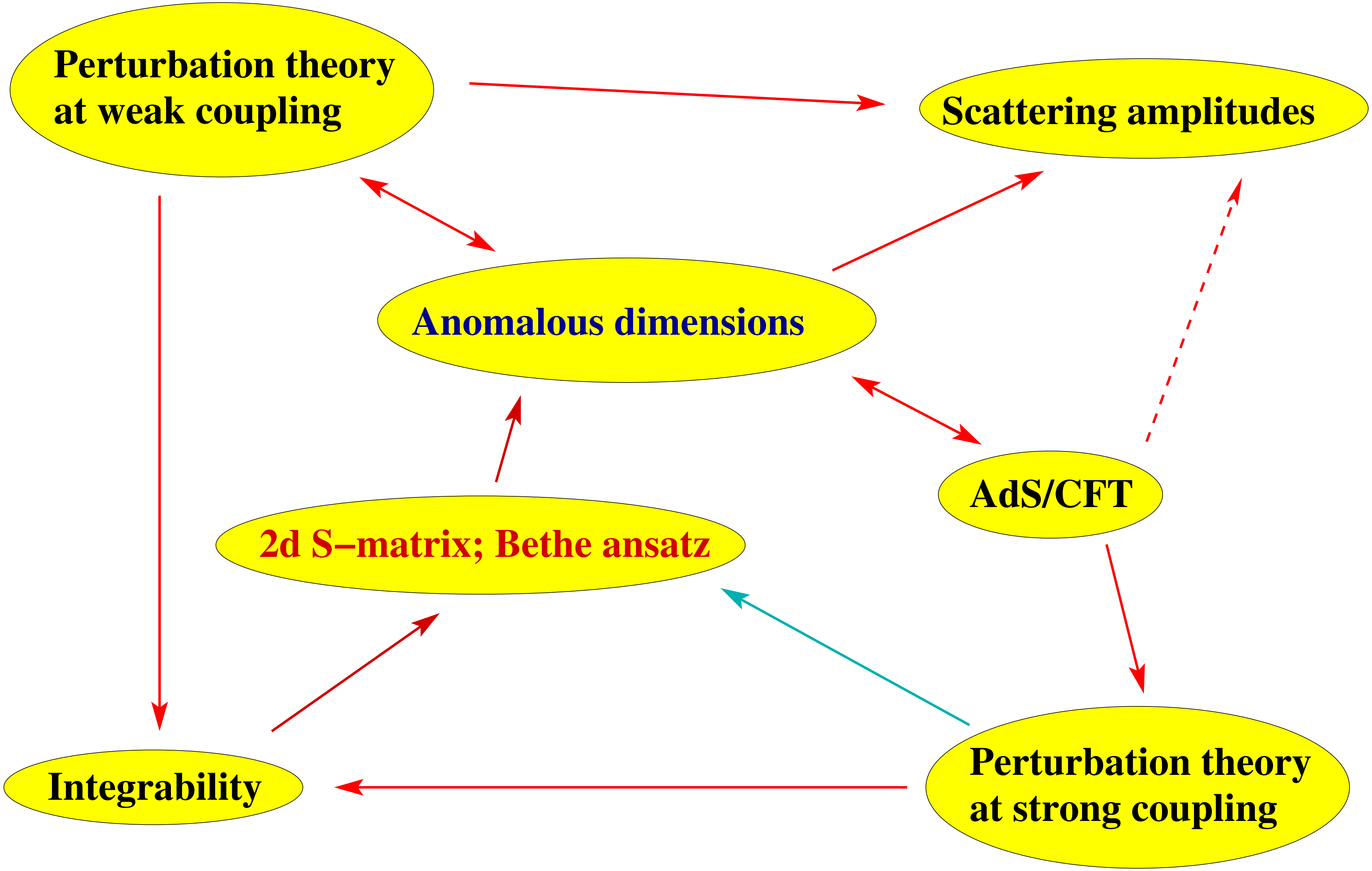
- O_n "natural" operators $\Rightarrow$ operator mixing – $\langle O_n(0) O_m(x)\rangle \neq \delta_{mn}$

  "good" operators = combinations of "natural" operators

◇ Clean formulation in terms of dilatation generator ($\mathcal{D} \in PSU(2,2|4)$)

$$\mathcal{D} O_n = \sum_m \gamma_{nm} O_m \quad \longrightarrow \quad \mathcal{D}\mathcal{O}_n = \Delta_n \mathcal{O}_n$$

In $\mathcal{N} = 4$ SYM the following relations appear to hold:



Anomalous dimensions, Integrability and such

- anomalous dimensions=eigenvalues of dilatation operator

$$D(\lambda) = D_1 + \lambda D_2 + \lambda^2 D_3 + \ldots$$

- (Long) String energies

$$E(\lambda) = \sqrt{\lambda} E_0 + E_1 + \frac{1}{\sqrt{\lambda}} E_2 + \ldots$$

- AdS/CFT: $E(\lambda) = D(\lambda)$

▪ Direct comparison difficult because of strong / weak coupling

◇ successful if effective small couplings exist (large q-numbers)

Integrability helps with the extrapolation

- nontrivial qualitative relations between unrelated operators

- Known explicit higher loop anomalous dimensions in $\mathcal{N}=4$ SYM:
  - BMN operators (2-loops) — Gross, Mikhailov, RR
  - single-magnon dispersion relation (all loops) — Gross, Mikhailov, RR; Santambrogio, Zanon
  - 2- and 3-loop cusp anomalous dimension — Kotikov, Lipatov, Onishchenko, Velizhanin; Bern, Dixon, Smirnov
  - 4-loop cusp anom. dim. tour de force — Bern, Czakon,Dixon, Kosower, Smirnov;
    - − further improvements — Cachazo, Spradlin, Volovich
- Using integrability in asymptotic regime – many more

  ◇ Various long operators at $\begin{cases} 1-\text{loop} \\ 2-\text{loops} \\ \text{higher} \end{cases}$

  - 1 − loop — Minahan, Zarembo;...
  - 2 − loops — Beisert, Kristjansen, Staudacher
  - higher — Beisert, Dippel, Staudacher

  ◇ conjecturally all and to all orders — Beisert, Hernandez, Lopez; Beisert, Eden, Staudacher

  ◇ remarkably: universal scaling function

  strong coupling NLO — Benna, Benvenutti, Klebanov, Scardicchio

  analytic approaches at LO and NLO — Kostov, Serban, Volin; Beccaria, De Angelis, Forini; Casteill, Kristjansen

  perturbative analytic; all orders — Basso, Korchemsky, Kotanski

Diagonalize: $D(\lambda)|\Psi\rangle = E|\Psi\rangle$ – make ansatz for eigenstate

$$|\Psi\rangle = \sum_{1\leq l_1<l_2\leq J} \Psi(l_1,l_2)|\ldots \underset{\substack{l_1\\\downarrow}}{Z\phi Z}\ldots \underset{\substack{l_2\\\downarrow}}{Z\phi Z}\ldots\rangle$$

- Bethe ansatz – asymptotic plane waves Bethe (1937)

$$\Psi(l_1,l_2) = e^{i(l_1p_1+l_2p_2)} + S(p_1,p_2,\lambda)e^{i(l_1p_2+l_2p_1)}$$

- Multi-particle: similar ansatz w/ $S(p_1,p_2) \mapsto S(p_1,p_2,p_3\ldots p_n)$

$S(p_1,p_2,p_3\ldots p_n)$ factorises in $S(p_i,p_j)$ factors; $E = \sum_i E(p_i)$

- Momenta p_i fixed from periodicity of wave function

Constraints on $S(p_1,p_2)$:

1. quantum-corrected symmetries + vacuum charge shift Beisert
2. factorization (YBE)
3. “crossing” symmetry Janik

- “Boundary conditions”: construct S from perturbative D

  ◇ interaction range smaller than operator length $\mapsto$ asymptotic BA

- symmetries+factorization $\mapsto$ up to overall function

Example:

$$\left(\frac{u_j+\sigma\frac{i}{2}}{u_j-\sigma\frac{i}{2}}\right)^L=\prod_{k=1}^{M}\frac{u_j-u_k+i}{u_j-u_k-i}e^{2i\theta(u_j,u_k)}$$

$$u(p)=\frac{1}{2}\cot\frac{p}{2}\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}$$

$$E-J=\sum_i\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_i}{2}}$$

  - θ − bilinear in higher conserved charges

- a curious consequence: features of θ common to all operators

- at 4-loops+ all anomalous dimensions are transcendental

Is this testable? Is the phase computable from first principles?

The phase:

◇ Universal to all sectors; general structure:

Arutyunov, Frolov, Staudacher; Beisert, Klose; Beisert, Tseytlin

$$\theta(u_1,u_2) = \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} \beta_{r,r+1+2\nu}(g)\Big[q_r(u_1)q_{r+1+2\nu}(u_2)-q_r(u_2)q_{r+1+2\nu}(u_1)\Big]$$

- strong coupling: $q \to \tilde{q}_r = g^{r-1}\, q_r \quad c_{r,s}(g) = g^{2-r-s}\beta_{r,s}(g){=}\sum_n c_{r,s}^{(n)} g^{1-n}$

$$c_{r,s}^{(n)} = \frac{(1-(-)^{r+s})(r-1)(s-1)}{2(-2\pi)^n\Gamma(n-1)} \frac{\Gamma(\frac{1}{2}(s+r+n-3))\Gamma(\frac{1}{2}(s-r+n-1))}{\Gamma(\frac{1}{2}(s+r-n+1))\Gamma(\frac{1}{2}(s-r-n+3))}$$

Beisert, Hernandez, Lopez

- weak coupling: $\beta_{r,r+1+2\nu}(g) = \sum_{\mu=\nu}^{\infty} g^{2r+2\nu+2\mu}\beta_{r,r+1+2\nu}^{(r+\nu+\mu)}$

• analytic continuation from strong coupling

Beisert, Eden, Staudacher

The phase:

◇ Universal to all sectors; general structure: Arutyunov, Frolov, Staudacher; Beisert, Klose; Beisert, Tseytlin

$$\theta(u_1,u_2) = \sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty} \beta_{r,r+1+2\nu}(g)\Big[q_r(u_1)q_{r+1+2\nu}(u_2) - q_r(u_2)q_{r+1+2\nu}(u_1)\Big]$$

- weak coupling: $\beta_{r,r+1+2\nu}(g) = \sum_{\mu=\nu}^{\infty} g^{2r+2\nu+2\mu}\beta_{r,r+1+2\nu}^{(r+\nu+\mu)}$

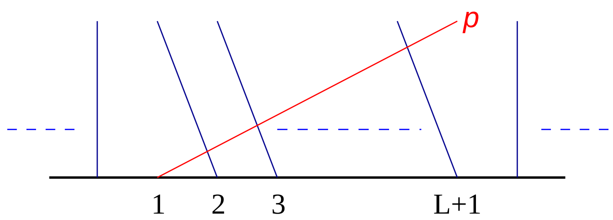
• analytic continuation from strong coupling – several possibilities: Beisert, Eden, Staudacher

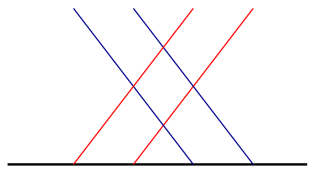
L	no $\zeta(2n+1)$	with $\zeta(2n+1)$
4	$\beta_{2,3}^{(3)} = 2\zeta(3)$	$\beta_{2,3}^{(3)} = 4\zeta(3)$
5	$\beta_{2,3}^{(4)} = -20\zeta(5)$	$\beta_{2,3}^{(4)} = -40\zeta(5)$
6	$\beta_{2,3}^{(5)} = 210\zeta(7)$,	$\beta_{2,3}^{(5)} = 420\zeta(7)$,
	$\beta_{3,4}^{(5)} = 12\zeta(5)$, $\beta_{2,5}^{(5)} = -4\zeta(5)$	$\beta_{3,4}^{(5)} = 24\zeta(5)$, $\beta_{2,5}^{(5)} = -8\zeta(5)$

- Find $D(\lambda)$ – renormalization: use off-shell Feynman rules
  - potential complications – the usual
  - main culprits – internal vector fields and fermions
    - 6 terms per gauge field vertex
    - many terms from fermion trace
- Can be avoided to some extent by an appropriate choice of states
  - symmetries and constraints go a long way
- Identify interactions that receive only scalar vertex contributions

Maximal shuffling:
- each vertex $\leftrightarrow$ permutation operator
- L-loops: interactions w/ L or $L+1$ sites
- no permutation of identical fields
- avoid states containing $\ldots(\phi Z)^n\ldots$

Examples:





- Constraints on the L-loop Hamiltonian in the $SU(2)$ sector:
  - $SU(2)$ symmetry and structure of Feynman diagrams
  - vacuum energy and the energy of other BPS states
  - dispersion relation
  - $[L/2]$-particle S-matrix $\leftrightarrow$ $L = 4$: need only 2-particle S-matrix
  - allow/introduce terms not affecting the eigenvalues
    - similarity transformations: $\mathcal{H}' = U^{-1}\mathcal{H}U$ & $U^\dagger \neq U^{-1}$

- past calculations:
  - spin chain Hamiltonian for AFS S-matrix — Beisert
    - imposes BMN scaling/existence of smooth continuum limit
  - 5-loop $SU(2)$ sector $\mathcal{H}$ assuming — Beisert, Dippel, Staudacher
    - integrability, smooth continuum limit, Feynman diagramatics
  - larger sectors

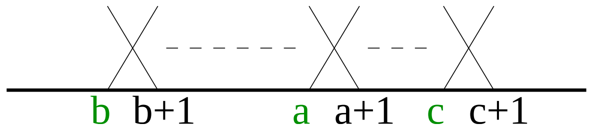
$$\mathcal{H}_0 = +\{\}$$

$$\mathcal{H}_1 = +2\{\} - 2\{1\}$$

$$\mathcal{H}_2 = -8\{\} + 12\{1\} - 2\big(\{1,2\} + \{2,1\}\big)$$

$$\begin{aligned}\mathcal{H}_3 = &+60\{\} - 104\{1\} + 4\{1,3\} + 24\big(\{1,2\} + \{2,1\}\big)\\ &- 4i\epsilon_2\{1,3,2\} + 4i\epsilon_2\{2,1,3\} - 4\big(\{1,2,3\} + \{3,2,1\}\big)\end{aligned}$$

$$\begin{aligned}\mathcal{H}_4 = &+\big(-560 - 4\beta_{2,3}\big)\{\}\\
&+ \big(+1072 + 12\beta_{2,3} + 8\epsilon_{3a}\big)\{1\}\\
&+ \big(-84 - 6\beta_{2,3} - 4\epsilon_{3a}\big)\{1,3\}\\
&- 4\{1,4\}\\
&+ \big(-302 - 4\beta_{2,3} - 8\epsilon_{3a}\big)\big(\{1,2\} + \{2,1\}\big)\\
&+ \big(+4\beta_{2,3} + 4\epsilon_{3a} + 2i\epsilon_{3c} - 4i\epsilon_{3d}\big)\{1,3,2\}\\
&+ \big(+4\beta_{2,3} + 4\epsilon_{3a} - 2i\epsilon_{3c} + 4i\epsilon_{3d}\big)\{2,1,3\}\\
&+ \big(4 - 2i\epsilon_{3c}\big)\big(\{1,2,4\} + \{1,4,3\}\big)\\
&+ \big(4 + 2i\epsilon_{3c}\big)\big(\{1,3,4\} + \{2,1,4\}\big)\\
&+ \big(+96 + 4\epsilon_{3a}\big)\big(\{1,2,3\} + \{3,2,1\}\big)\\
&+ \big(-12 - 2\beta_{2,3} - 4\epsilon_{3a}\big)\{2,1,3,2\}\\
&+ \big(+18 + 4\epsilon_{3a}\big)\big(\{1,3,2,4\} + \{2,1,4,3\}\big)\\
&+ \big(-8 - 2\epsilon_{3a} - 2i\epsilon_{3b}\big)\big(\{1,2,4,3\} + \{1,4,3,2\}\big)\\
&+ \big(-8 - 2\epsilon_{3a} + 2i\epsilon_{3b}\big)\big(\{2,1,3,4\} + \{3,2,1,4\}\big)\\
&- 10\big(\{1,2,3,4\} + \{4,3,2,1\}\big)\end{aligned}$$

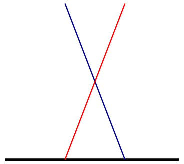


- $\{...cba\} = ...P_{c,c+1}P_{b,b+1}P_{a,a+1}$
- β=phase parameters
- ϵ=similarity parameters

  $\mathcal{H} \to U(\epsilon)^{-1}\mathcal{H}U(\epsilon)$

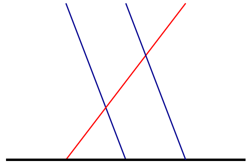
- Maximal shuffling interactions at 1-, 2-, 3- and 4-loops

1-loop :



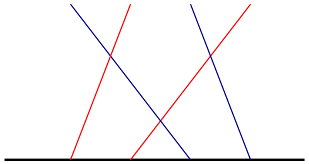
$\{1\}$

2-loops :

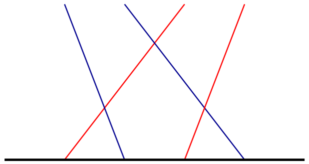


$\{2,1\}$

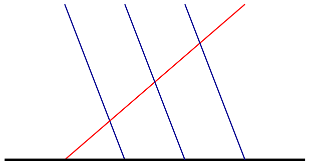
3-loops :



$\{1,3,2\}$

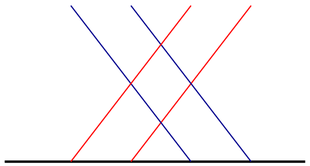


$\{2,1,3\}$

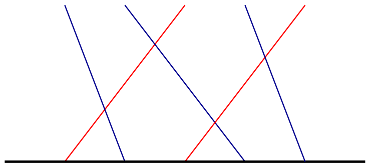


$\{3,2,1\}$

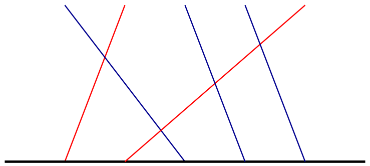
4-loops :



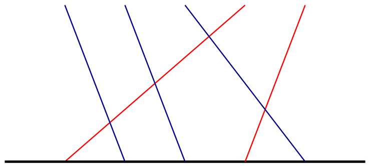
$\{2,1,3,2\}$



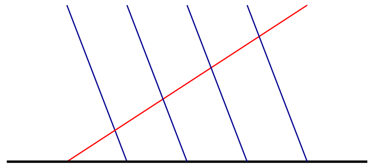
$\{2,1,4,3\}$



$\{1,4,3,2\}$



$\{3,2,1,4\}$



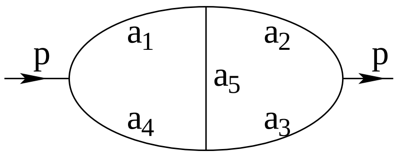
$\{4,3,2,1\}$

- The calculation:

  - regularization of IR divergences: keep some legs off-shell

  - reduce to "master integrals"

    ◇ successive evaluation of 1-loop bubbles and use of partial integration reduces everything to 1- and 2-loop integrals



with $a_{1,2,3} = 1$, $a_4 = 1 + n\,\epsilon$, $a_5 = m\,\epsilon \quad m, n \in \mathbb{Z}$

- Subtraction:

  counterterms/R-operation $\qquad \gamma = \lim_{\epsilon \longrightarrow 0} \epsilon \mathbf{Z}^{-1} \frac{d\mathbf{Z}}{d \ln g_{YM}}$

  ◇ equivalent recursive subtraction rules

Beisert, Kristjansen, Staudacher

- Coefficients of maximal shuffling terms: Beisert, McLoughlin, RR

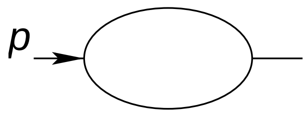
$$
\begin{array}{rl}
(-2)\,\{1\} & (-\ \ 4+\ \ 4\,\zeta(3))\{2,1,3,2\} \\
(-2)\,\{2,1\} & (+10-12\,\zeta(3))\{2,1,4,3\} \\
(+4)\,\{1,3,2\} & (+\ \ 2+\ \ 8\,\zeta(3))\{1,4,3,2\} \\
(-4)\,\{2,1,3\} & (-10+\ \ 4\,\zeta(3))\{3,2,1,4\} \\
(-4)\,\{3,2,1\} & (-10)\{4,3,2,1\}
\end{array}
$$

- phase and similarity trasformation coefficients

$$\beta_{2,3}^{(3)} = 4\zeta(3)$$

$$i\epsilon_2 = -1, \qquad \epsilon_{3a} = -2 - 3\zeta(3), \qquad i\epsilon_{3b} = -3 - \zeta(3)$$

- Could/Should this have been expected?

$L(1,1) \propto$  $\propto \frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)^2}{\Gamma(d-2)}$

$$= \frac{\left(4\pi e^{-\gamma}\right)^{\epsilon}}{16\pi^2\epsilon}\left(1+2\epsilon+\left(4-\frac{1}{12}\pi^2\right)\epsilon^2+\left(8-\frac{1}{6}\pi^2-\frac{7}{3}\zeta(3)\right)\epsilon^3+\ldots\right)$$

- $\zeta(3)$ can appear in an anom. dim. iff additional $1/\epsilon^3$ is present

$\longrightarrow$ 4-loops!

Prospects at higher loops

- Higher-loop cusp extractable from 4-gluon scattering
  - 5-loop integrand known — Bern, Carrasco, Johansson, Kosower
  - extraction of the $1/\epsilon^2$ pole appears challenging

- Direct extraction of dressing phase from $SU(2)$-sector Hamiltonian
  - ◇ simpler integrals – lower-point functions
  - ◇ algebraic complications in relating Hamiltonian and phase
  - ◇ proof of integrability requires more than maximal shuffling

▪ Assuming integrability:

◇ only leading order term in the expansion of each phase coefficient is computable from maximal shuffling interactions

$\theta = \theta(\beta_{r,s}(\lambda)) \Rightarrow$ Hamiltonian depends on complete dressing phase coefficients $\beta_{r,s}(\lambda)$ rather than $\beta_{r,s}^{(l)}$

- subleading terms in expansion of β involve vectors and spinors

- test: all 8 5-loops maximal shuffling terms free of $\zeta(5)$

McLoughlin, RR

• How much to expect: look again at the expansion of $L(1,1)$:

$$L(1,1) \sim \frac{1}{\epsilon}\left(\dots\zeta(2n+1)\left(a_{2n+1}\epsilon^{2n+1} + b_{2n+2}\epsilon^{2n+2}\right) + \dots\right)$$

$\rightarrow$ expect $\zeta(2n+1)$ at $L = 2n+2$ in maximal shuffling interactions

- some tests:

$$\mathcal{H}_6 = \dots + (18 - 8\zeta(3) - 4\zeta(5))\{2,1,3,2,4,5\} + \dots$$
$$\mathcal{H}_8 = \dots - (19+10\zeta(3)-9\zeta(3)^2+24\zeta(5) - \zeta(7))\{2,1,3,2,4,5,6,7\} + \dots$$

Summary

- Proved integrability of the dilatation operator at four loops

- Four-loop anomalous dimensions in the $SU(2)$ sector consistent with conjectured BES/BHL dressing phase

  - Dilatation operator has transcentental coefficients consistent with the existence of interpolating functions

- Leading terms in the coefficients of the dressing phase are easily accessible to direct Feynman diagram calculations

  - related to maximal shuffling interactions

  - some contributions with obvious recursive structure