A phase transitions in Wilson loops in QCD at infinite number of colors.

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In nonabelian SU(N) YM, at infinite N, the eigenvalue distribution of a Wilson loop undergoes a nonanalytic change as the loop is dilated. For small loops the density of eigenvalue close to -1 is exactly zero, and for large loops it is larger than zero. The region close to the critical scale is universal and the universality class is the same in 2,3,4 dimensions. This claim is provable in 2d by analytical means and can be checked by numerical methods in 3d and 4d.

Introduction I

2 reasons for YM $SU(N)$ ⟺ string theory:

1. "Physical string": tubular region of elevated chromo-electric energy connecting stationary sources in $N, \bar{N}$: confinement needed.
2. Wilson loop operators W($\mathcal{C}$) are the basic (non-local gauge invariant) d.o.f. indicating

$$(x_1, x_2, x_3, x_4)(\text{points}) \rightarrow \mathcal{C}(\text{curves})$$

Inroduction II

- AdS/CFT: no physical strings needed for a string description; hence reason 2) seems better.
- 2) connects to strings via MM loop equations. There is a problem in the UV, easier in SUSY.
- Large N ala ‘t Hooft simplifies loop equations, keeps planar Feynman diagrams, and infinite N seems to be a free string theory, which amounts to a special unknown 2d sigma model.

Introduction III

- This work is on the field theory side.
- Use lattice field theory to find a large N phase transition in Wilson loops under dilatation.
- The transition implies that the 1/N series for loops of critical size l_c diverges: handle condensation in terms of Feynman diagrams ?
- On the string side, an ***effective 2d Lagrangian,*** depending just on $x_\mu(\sigma,\tau)$, and expanded in 1/l, breaks down.

Introduction IV

- What happens at $l=l_c$ should be understandable since the large N transition is of identical type in d=2,3,4 Euclidean dimensions.
- My talk will explain what happens on the field theory side in d=2 (analytical) and in d=3 (numerical).

Main Claim

The eigenvalue distribution associated with a simple Wilson loop at infinite N undergoes a phase transition of universal character as the loop is dilated.The phase transition occurs when the eigenvalue distribution changes from having finite support to covering the entire unit circle. The transition persists in the continuum limit for properly regulated Wilson loops. For these operators, the short distance – long distance crossover occurs over a scale range that shrinks to zero as N increases. There is no transition in the traces of finitely wound Wilson loops.

The phase transition

- Below the transition, the tails of the eigenvalue distribution at -1 disappear, and there is a gap there. For finite N the tails are exponentially suppressed.
- There exists a large N universality class governing the large N phase transition that happens at the critical scale. That universality class has exactly soluble representatives and applies to QCD in two three and four Euclidean dimensions.

2d defines the large N universality class

A general formulation of the universality class is in terms of a random multiplicative ensemble: Consider n independent SU(N) matrices i.i.d. with a measure dμ. A W-loop of area n is represented by the product of the unitary matrices, in any order and evolves in n by diffusion on SU(N) starting from a point source. This is exactly true in 2d. Thus, this is the DURHUUS - OLESEN universality class and is an exact version of the old conjecture of "dimensional reduction".

Eigenvalue distribution

The average characteristic polynomial of W captures the essential features of the collective behavior of the W eigenvalues. We are interested in the region around eigenvalue –1. t is the 2D unitless area -- critical at 4.

$$Q_N(z,t) \equiv \langle \mathrm{det}(z - W) \rangle$$

$$\text{Scaling: } z = -e^y, \frac{1}{t} = \frac{1}{4} + a$$

A representation of the 2D $Q_N(z,t)$

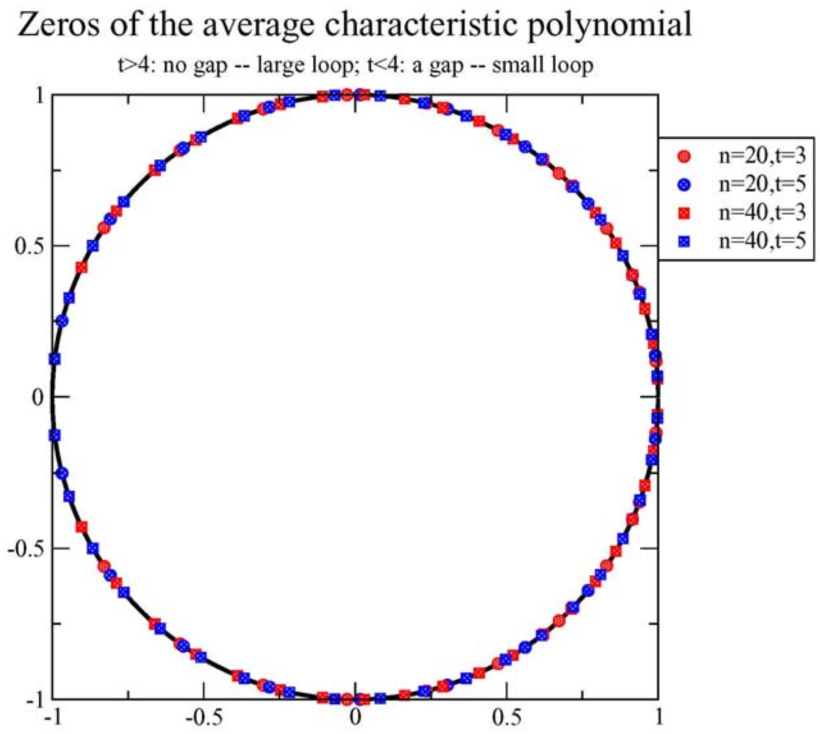
$$\langle \det(z-W)\rangle = \int_0^\infty da \left(z - ae^{-t/2}\right)^N P(a,t)$$

where the probability distribution $P(a,t)$ satisfies a Fokker-Planck equation:

$$N\partial_t P(a,t) = \partial_a(aP) + \frac{1}{2}\partial_a a^2 \partial_a P$$

The large N limit is determined by balancing the two factors, since $P \sim \exp(-N \log^2 a/(2t))$. Integrating exactly over a one finds the 2d $Q_N(z,t)$. Extracting some factors from $Q_N(z,t)$ leaves an expression invariant under $z \to 1/z$.

Q_N (z,t) and the eigenvalue distribution



By Lee-Yang we can prove that all the zeros of the average characteristic polynomial $Q_N(z,t)$ lie on the unit circle. At large N the zeros have statistical properties similar to those of the eigenvalues of W.

Large N finite size scaling

In the scaling limit take $N \to \infty$ keeping scaling variables fixed:

$$y = \frac{2}{6^{1/4} N^{3/4}} \xi, \quad a = \frac{1}{4\sqrt{3N}} \alpha$$

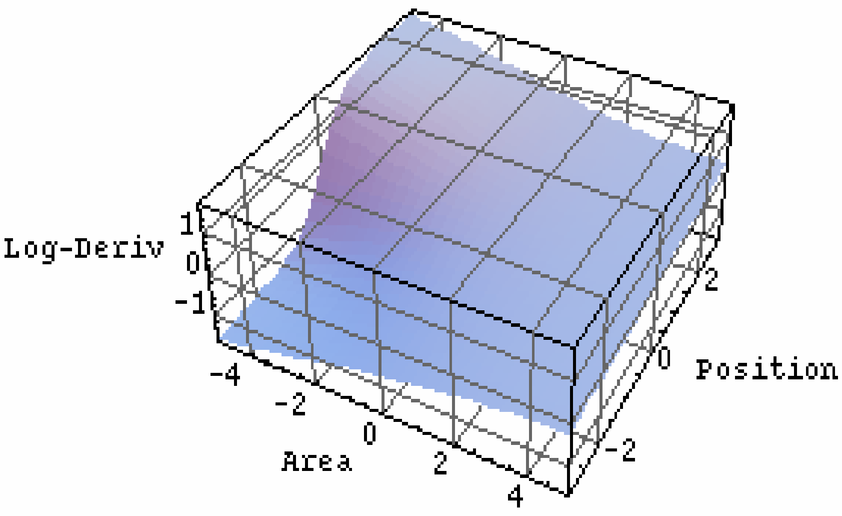
Q_N is proportional to a partition function

$$\int_{-\infty}^{\infty} du e^{-u^4 - \alpha u^2 + \xi u} \quad \text{even in } \xi$$

which is a generalized Airy function. This formula, with different numbers, but identical powers of N, is expected to hold in all dimensions. Derivatives w.r.t. ξ generate α dependent moments $\langle u^{2k} \rangle$.

Scaling Function

At infinite N the log derivative of Q_N is a 2D electric field associated with a static charge distribution representing the eigenvalues of W. When there is no gap there is a jump in the field at –1 on the unit circle. When there is a gap there is no jump. All this gets smoothed out at finite N in a universal manner.



Amplitudes I

In any dimensions we define

$$O_N(y,l) = \langle \det(e^{y/2} + e^{-y/2}W) \rangle$$

O_N is even in y. Expand $O_N(y,l)$ around $y = 0$:

$$O_N(y,l) = C_0(l,N) + C_1(l,N)y^2 + C_2(l,N)y^4 + ...$$

We eliminate non-unviersal quantiites by looking at:

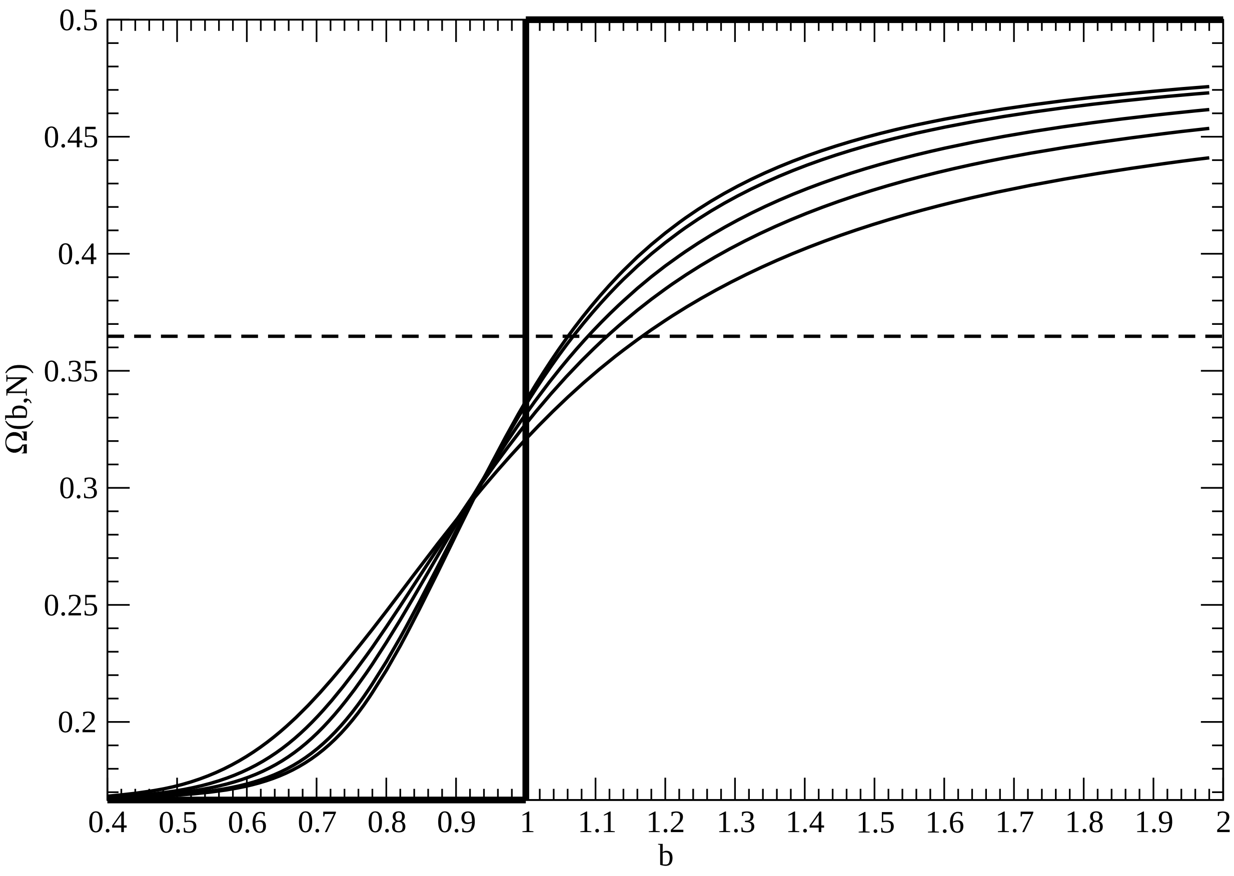
$$\Omega(l,N) = \frac{C_0(l,N)C_2(l,N)}{C_1^2(l,N)}$$

Amplitudes II

$$\Omega(l,N) = \begin{cases} \Gamma^4(\frac{1}{4})/48\pi^2, & \text{for } l = l_c; \\ 1/6, & \text{for } l >> l_c; \\ 1/2, & \text{for } l << l_c. \end{cases}$$

This is used to find l_c. There are now two main amplitudes associated with $N^{-3/4}$ and with $N^{-1/2}$ measuring deviation from the critical point at $y = 0$ and $l = l_c$. The asymptotic values $1/6, 1/2$ correspond respectively to a single gaussian and a double gaussian distribution of u.

Ω (l,N)



The lines represent various values of N, from 17 to 47. On the right, N=47 is the top line: $b \sim 1/l^2$

Amplitudes III

The universal parameters α and ξ in the generalized Airy function are related to Ω

$$\alpha = a_2(l/l_c - 1)\sqrt{3N}$$

$$d\Omega/dl|_{l=l_c} = r_0\sqrt{3N}a_2$$

$$\xi = (3N^3/4)^{1/4} y a_1$$

$$C_1/C_0 = r_1(3N^3/4)^{1/2} a_1^2$$

where, r_0, r_1 are known numbers. a_1, a_2 are the amplitudes needed to bring the observale O_N in the general case into the canonical form.

Three dimensions

The numerical prefactors are such that in 2d $a_1 = a_2 = 1$ if $l = 4/b$, where b is the 't Hooft gauge coupling. In 3d we calcuale using Monte Carlo the values of $a_{1,2}$ as defined above and show numerically that they have finite large N and continuum limits. I skip the technical details of how the extrapolations are done, but their consistency constitutes numerical proof that 3d is in the same universality class as 2d.

Dimensional reduction

If one matches 2d to 3d by expressing everything in terms of the string tension the amplitudes a_1 and a_2 come out close numerically. This is a reflection of the ~25 years old ideas and observations of Ambjorn, Greensite, Olesen, Makeenko and others that two dimensional YM Wilson loops match surprisingly well their 3d and 4d counterparts.

Final message

- The numerical results in 3D are in agreement with large N universality.
- The large N phase transition can occur also in a non-confining YM theory.
- This transition is not seen in traces of finitely wound Wilson loops.
- The Wilson loop characteristic polynomial is an interesting observable.