Strings and non-compact spaces

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Introduction

String background $M^r \times I^s \rightarrow N^r \times I^s$

M^r *Minkowski* $\rightarrow N^r$ *General non-compact* $(1,r\text{-}1)$

- General understanding of string and *M*-theory.
- *AdS/CFT* duality (N^r is an *AdS* space).
- Nice and interesting mathematics (non-compact affine Lie algebras).

Gauged WZNW models

- Non-compact part: Gauged WZNW model G/H, G is a non-compact group and H is a sub-group.
- G has p non-compact and q compact dimensions, H has p' non-compact and q-1 compact $\Rightarrow$ G/H has structure $(1,p-p')$.
- World-sheet formulation with exact conformal invariance.
- Bosonic and world-sheet susy.
- Here: Unitarity and 1-loop partition function

G/H spaces with $(1,d)$ structure

G	H	# G Generators		# H Generators		Signature
		compact	non compact	compact	non compact	
$SU(p,q)$	$SU(p)\times SU(q)$	p^2+q^2-1	$2pq$	p^2+q^2-2	0	$(1,2pq)$
$SO(p,2)$	$SO(p,1)$	$\frac{1}{2}p(p-1)+1$	$2p$	$\frac{1}{2}p(p-1)$	p	$(1,p)$
$SO(p,2)$	$SO(p)$	$\frac{1}{2}p(p-1)+1$	$2p$	$\frac{1}{2}p(p-1)$	0	$(1,2p)$
$Sp(2p,\mathbb{R})$	$SU(p)$	p^2	$p(p+1)$	p^2-1	0	$(1,p(p+1))$
$SO^*(2p)$	$SU(p)$	p^2	$p(p-1)$	p^2-1	0	$(1,p(p-1))$
$E_{6(-14)}$	$SO(10)$	46	32	45	0	$(1,32)$
$E_{7(-25)}$	$E_{6(-78)}$	79	54	78	0	$(1,54)$

Table 2: Coset spaces G/H with only one time coordinate (for simple groups G)

From Ginsparg & Quevedo, NBP 385, 527 (1992)

$G=SL(2,\mathbf{R})$ (or coverings thereof), $H=1$

Many properties have been worked out:

- Unitarity

  (Dixon, Lykken & Peskin '89; SH '91 '92; Henningson & SH '91; Evans, Gaberdiel & Perry '98; Maldacena & Ooguri '01)

- New sectors of states: Spectral flow

  (Henningson,SH, Roberts & Sundborg '91; Maldacena & Ooguri '00; Hikida,Hosomichi & Sugawara '00; Argurio, Giveon &Shomer '00)

- Correlation functions

  (Teschner '99,'00; Giribet & Nunez '01; Maldacena & Ooguri '01; Satoh '02; Hofman & Nunez '04; Ribault '05; Herscovich, Minces & Nunez '05;…)

- *AdS/CFT* tests

  (Maldacena & Ooguri '01,'02; Gaberdiel & Kirsch '07; Dabholkar & Pakman '07;….)

$SL(2,\mathbf{R})$ cont.

- Unitarity: *k<-2* (⇒ *c≥ 3),* discrete hw or lw repr's (k+1)/2<*j<-1/2*, principle continous repr's.
- Strings that stretch and wind, spectral flow:

$$J_n^{\pm} \longrightarrow J_{n\pm s}^{\pm} \;\; ; \;\; J_n^3 \longrightarrow J_n^3 + \frac{k}{2} s\delta_{n,0} \;\; ; \;\; s = integer$$
$$L_n \longrightarrow L_n + sJ_n^3 + \frac{k}{4} s^2 \delta_{n,0}$$

- Spectrally flowed states needed for consistent interactions and *AdS/CFT* correspondence.

$AdS_{(d,1)}$ spaces

- Construction: $SO(d,2)/SO(d,1)$
- $SO(d,2)$ has unitary discrete hw repr's and continous unitary repr's.. For $SO(d,1)$ all unitary repr's are continous.
- "Time" is highly non-trivially embedded $\Rightarrow$ difficult to analyze unitarity.
- $SO(d,2)$ has another interesting coset: $SO(d,2)/SO(d) \rightarrow$ Hermitian symmetric spaces $\rightarrow$

???

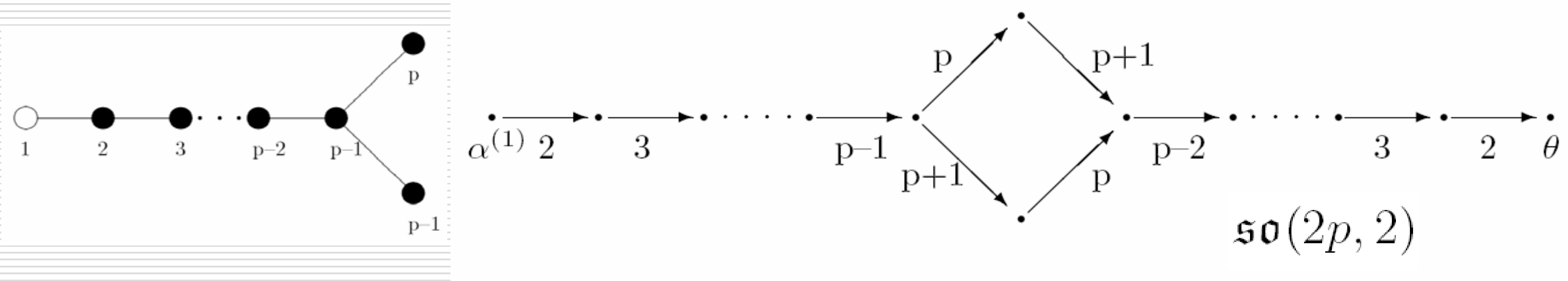
$$SO(d) \subset SO(d,1) \implies \frac{SO(d,2)}{SO(d,1)} \subset \frac{SO(d,2)}{SO(d)}$$

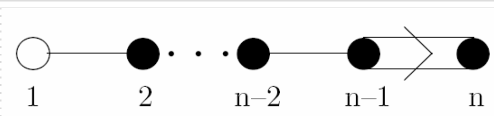
Information about *AdS* through the Hermitian symmetric case (necessary conditions ?).

Hermitian symmetric series G/H

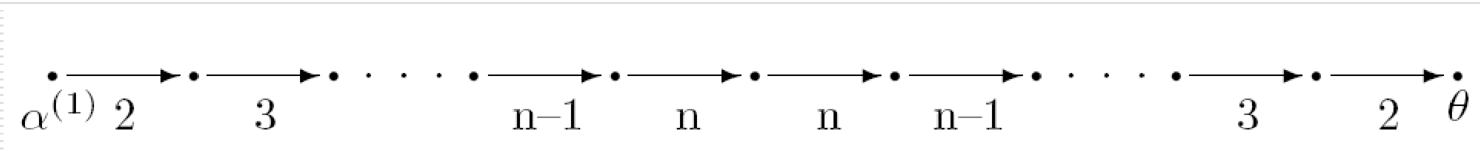
- ”Simple” embedding of time: max compact subgroup $H' = H \times Z(H')$, time is associated with the center $Z(H')$ (or coverings thereof).
- Structure of non-compact roots completely known. *(Jakobsen ’83)* $\rightarrow$

Structure of non-compact roots for *so(d,2)*



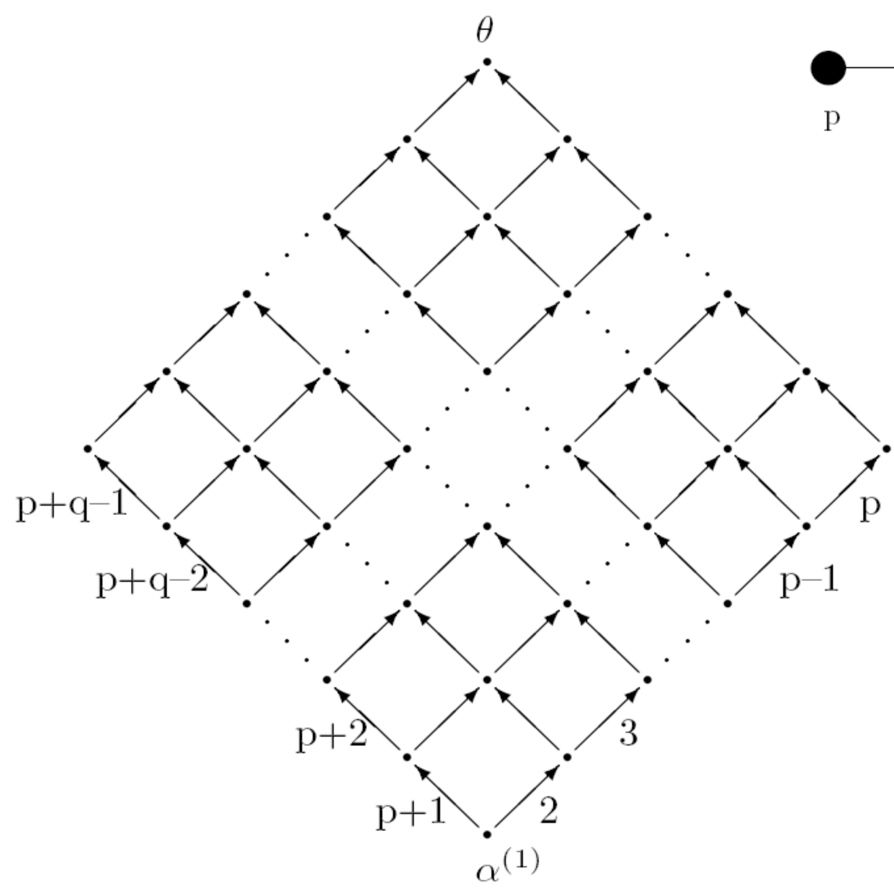


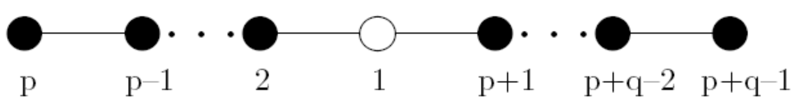
$\mathfrak{so}(2p-1,2)$



Choice of basis: Unique simple non-compact root
(from Jakobsen, J.Funct.Anal 52 (1983) 385)

Non-compact roots for *su(p,q)*





Hermitian symmetric series G/H

- ”Simple” embedding of time: max compact subgroup $H' = H \times Z(H')$, time is associated with the center $Z(H')$ (or coverings thereof).
- Structure of non-compact roots completely known. *(Jakobsen ’83)*
- Only series for the non-compact case with unitary discrete highest weight representations. All such repr’s classified. (Jakobsen ’83; Enright, Howe & Wallach ’83)
- Complete series $G = SU(p,q)\ (p,q \geq 1)$, $SO(p,2)$, $SO^*(2n)$, $Sp(2n,\mathbf{R})$, $E_{6|-14}\, E_{7|-25}$

G/H spaces with *(1,d)* structure

	G	H	# G Generators		# H Generators		Signature
			compact	non compact	compact	non compact	
HSS	$SU(p,q)$	$SU(p)\times SU(q)$	p^2+q^2-1	$2pq$	p^2+q^2-2	0	$(1,2pq)$
AdS	$SO(p,2)$	$SO(p,1)$	$\frac{1}{2}p(p-1)+1$	$2p$	$\frac{1}{2}p(p-1)$	p	$(1,p)$
HSS	$SO(p,2)$	$SO(p)$	$\frac{1}{2}p(p-1)+1$	$2p$	$\frac{1}{2}p(p-1)$	0	$(1,2p)$
HSS	$Sp(2p,\mathbb{R})$	$SU(p)$	p^2	$p(p+1)$	p^2-1	0	$(1,p(p+1))$
HSS	$SO^*(2p)$	$SU(p)$	p^2	$p(p-1)$	p^2-1	0	$(1,p(p-1))$
HSS	$E_{6(-14)}$	$SO(10)$	46	32	45	0	$(1,32)$
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Discrete highest weight representations

- Discrete unitary hw-repr's of G must have unitary hw of H and of every $SU(1,1)$ subgroup in G. Repr's characterized by the value of $Z(H')$ $[H'=H \times Z(H)$ eg $G=SO(d,2), H'=SO(d) \times Z]$.
- No "natural" affine generalization of unitary discrete hw repr's of G: $su(2) \to su(2)_k$, $k \geq 0$; $su(1,1) \to su(1,1)_k$, $k<-2$
- Unitarity in classical limit $|k| \to \infty$ implies $k<0$
- Choose another class: Antidominant hw repr's antidominant affine weights

$\hat{\mu}$: $(\hat{\mu}+\hat{\rho},\hat{\alpha})<0$ ⟹ $k<(\theta,\mu)-1$ $\quad k<-g^{\vee}$

cf. dominant affine weights $\hat{\mu}$: $(\hat{\mu},\hat{\alpha})\geq 0$

GKO coset construction

- Goddard-Kent-Olive (GKO) construction ('85): Impose hw and Virasoro conditions:

$$J^{+}_{(H)}\,|\Phi\rangle = 0 \qquad L_n\,|\Phi\rangle = (L_0 - 1)\,|\Phi\rangle = 0$$

- Fails to give unitarity by counterexamples: *G=su(2,1) , H=SU(2); n≥ –(μ²+1)*

$$|\Phi\rangle = \left[J_0^{-(\alpha^{(1)}+\alpha^{(2)})}\left(J_0^{-\alpha^{(1)}}\right)^n + \frac{1}{\mu^2}\left(J_0^{-\alpha^{(1)}}\right)^{n+1} J_0^{-\alpha^{(2)}}\right]\left|0;\mu^1,\mu^2\right\rangle$$

BRST coset construction

- BRST construction of Karabali & Schnitzer ('90): Introduce an auxiliary H sector, ghosts and a BRST charge and condition

$$Q^{G/H} = \left(J_{(H)} + \tilde{J}_{(H)} \right) c_{(H)} + (\ldots) b_{(H)} c_{(H)} c_{(H)}$$

$$Q = Q^{G/H} + Q^{\mathrm{V}ir}$$

$$Q \left| \Phi \right\rangle = b_0^i \left| \Phi \right\rangle = \mathcal{P}_0 \left| \Phi \right\rangle 0$$

Gives unitarity i.e. BRST and GKO constructions are inequivalent!

GKO vs BRST

$$J^{+}_{(H)}\,|\Phi\rangle = 0 \longrightarrow Q^{G/H}\,|\Phi\rangle = 0$$

$$\nearrow \quad |\Phi\rangle = Q^{G/H}\,|\Psi\rangle$$

$$\searrow \quad |\Phi\rangle \neq Q^{G/H}\,|\Psi\rangle$$

- Auxiliary H sector is crucial for decoupling of non-unitary states
- The lack of complete reducibility w.r.t. H is ”responsible” for GKO breakdown.

Unitarity

- Antidominant hw repr μ for G-sector $k < (\theta, \mu) - 1$ dominant hw repr for auxiliary H-sector (i.e. unitary).
- Requires highest weights to be integer valued in compact directions.
- The level k is quantized kI *is* integer (I is the embedding index=1 or 2) $\Rightarrow$ limited values of conformal anomaly c

$$c = c_{(G)} - c_{(H)};$$

$$c_{(G)} = \frac{Dk}{k + g^\vee}\ , \quad k < -g^\vee$$

- Unitarity also proven for G/H' conformal field theories (i.e. "time" direction divided out, no Virasoro condition imposed).

Possible values of c

$\mathfrak{so}(p,2)/\mathfrak{so}(p)$			
p	c_{max}	k_{max}	c_{min}
3	21.2500	-5	7
4	22	-8	9
5	24.7500	-11	11
6	24.8000	-16	13
7	25.6235	-22	15
8	25.9322	-16	17
9	25.9168	-46	19
10	25.9377	-31	21
11	25.9908	-135	23
12	25.9977	-448	25

$\mathfrak{su}(p,1)/\mathfrak{su}(p)$			
p	c_{max}	k_{max}	c_{min}
2	26	-4	5
3	21	-7	7
4	23	-10	9
5	26	-13	11
6	24.8462	-19	13
7	25.9804	-25	15
8	25.6667	-36	17
9	25.8272	-52	19
10	25.9876	-80	21
11	25.9826	-148	23
12	25.9992	-485	25

$$k = k_{\mathrm{max}},\ \ k < -g^{\vee} \Rightarrow c_{\mathrm{min}}$$

$$c \leq 26 \Rightarrow c_{\mathrm{max}}$$

Generalized branching functions

- Compact G, unitary repr's *branching function*

$$\chi_\mu^G = \sum_\lambda B_{\mu,\lambda}^{G/H} \chi_\lambda^H$$

- True also for affine compact case for unitary repr's (Kac '74) .
- Not in general true for non-compact affine case.
- Generalized branching functions *BRST inv.*

$$\mathcal{B}_{\mu,\lambda'}^{G/H}(q,\phi) \equiv \left[\chi_\mu^G(q,\phi,\theta)\chi_{\lambda'}^{H,\mathrm{aux}}(q,\theta)\chi^{ghost}(q,\theta)\right]_{\theta-indep}$$

- For compact affine case one may prove for unitary repr's (SH & Rhedin '94)

$$B_{\mu,\lambda}^{G/H} = \mathcal{B}_{\mu,-\lambda-2\rho}^{G/H}$$

1-loop partition function

$$
\begin{aligned}
\mathcal{B}^{(\mathfrak{g},\mathfrak{h}')}(q,\phi) &= [q^{[\frac{c_2^{\mathfrak{g}}}{2\left(k+g_{\mathfrak{g}}^{\vee}\right)}-\frac{c_2^{\tilde{\mathfrak{h}}'}}{2\left(\kappa+g_{\mathfrak{h}'}^{\vee}\right)}-1]} \exp\left[\mathrm{i}\left(\mu_{\|}+\tilde{\mu}+2\rho_{\mathfrak{h}'},\theta\right)+\mathrm{i}\phi\mu_{\perp}\right] \\
&\times \prod_{\alpha\in\Delta_n^+} \frac{1}{1-\exp\left[-\mathrm{i}\phi\right]\exp\left[-\mathrm{i}\left(\theta,\alpha_{\|}\right)\right]} \\
&\times \prod_{m=1}^{\infty}\prod_{\alpha\in\Delta_n} \frac{1}{1-q^m\exp\left[-\mathrm{i}\phi\right]\exp\left[-\mathrm{i}\left(\theta,\alpha_{\|}\right)\right]} \prod_{m=1}^{\infty}\frac{1}{(1-q^m)} \\
&\times \prod_{m=1}^{\infty}\prod_{\alpha\in\Delta_n} \frac{1}{1-q^m\exp\left[\mathrm{i}\phi\right]\exp\left[-\mathrm{i}\left(\theta,\alpha_{\|}\right)\right]} \prod_{m=1}^{\infty}\frac{1}{(1-q^m)} \\
&\times \sum_{w\in W(\mathfrak{h}')} (-1)^{\mathrm{sign}(w)} \sum_{\bar{\beta}\in L^{\vee}} \left[\exp\left[\mathrm{i}\left(w\left(\tilde{\mu}+\rho_{\mathfrak{h}'}+\bar{\beta}\left(\tilde{\kappa}+g_{\mathfrak{h}'}^{\vee}\right)\right),\theta\right)\right]\right. \\
&\times q^{\left(\bar{\beta},\tilde{\mu}+\rho_{\mathfrak{h}'}\right)+\frac{1}{2}\left(\bar{\beta},\bar{\beta}\right)\left(\tilde{\kappa}+g_{\mathfrak{h}'}^{\vee}\right)}]_{\theta-indep}
\end{aligned}
$$

- 1-loop partition function can be found from $\phi \to 0$ limit.

Supersymmetric case

- World-sheet supersymmetric construction of Kazama and Suzuki ('89)
- Fermionic fields decouple after field redefinitions, yields new currents with $k \to k - g^\vee$
- The coset construction uses BRST approach (unshifted fields)

$$j^1_{BRST} \;=\; :c_a\left(J^a + \tilde{J}^a\right): + \gamma_a\left(\lambda^a + \tilde{\lambda}^a\right) - \frac{1}{2} f^{ab}{}_c c_a c_b b^c + f^{ab}{}_c c_a \gamma_b \beta^c,$$

- Unitarity proven for antidominant hw weights. Necessary conditions are kI integer, integer valued weights in compact directions.
- Generalized branching functions derived.

Possible values of *c (prel.)*

$\mathfrak{su}(p,1)/\mathfrak{su}(p)$

p	c_{max}	k_{max}	c_{min}	dim
2	13.5	-3	7.5	5
3	15	-6	10.5	7
4	15	-40	13.5	9

$\mathfrak{so}^*(6)/\mathfrak{su}(3)$

p	c_{max}	k_{max}	c_{min}	dim
2	15	-6	10.5	7

$\mathfrak{su}(p,2)/(\mathfrak{su}(p)\oplus\mathfrak{su}(2))$

p	c_{max}	k_{max}	c_{min}	dim
2	15	-32	13.5	9

$\mathfrak{so}(4,2)/\mathfrak{so}(4)$

p	c_{max}	k_{max}	c_{min}	dim
2	15	-32	13.5	9

$\mathfrak{sp}(4,\mathbb{R})/\mathfrak{su}(2)$

p	c_{max}	k_{max}	c_{min}	dim
2	15	-6	10.5	7

$AdS_{(d,1)}$ spaces

- ☺ Real forms, regular embeddings and hw repr's *(d=4)*.
- ☺ Construction of explicit regular embedding $so(d) \subset so(d,1) \subset so(d,2)$ for *d even* and construction of BRST charge.
- ☺ Derivation of relevant characters and branching function.
- ☹ Need to know number of negative normed states (signature function for $so(d,1)$).

Further work

- Unitarity for *AdS* spaces for discrete representations.
- Other representations (other discrete & continous).
- Spectral flow sectors of states.
- Correlation functions (tree and loop).
- Space-time supersymmetry.
- *AdS/CFT* duality.

Summary

- Unitarity proven for anti-dominant hw-repr's of G in bosonic and world-sheet susy cases.
- GKO construction fails. BRST construction works.
- Unitarity for integer valued highest weights and integer values of kI.
- Generalized branching functions constructed.
- Well on the way for the *AdS* case.

Thank you for your attention!