Thank you for the invitation.

RELATIONS BETWEEN STANDARD MODEL

PARAMETERS FROM T' SYMMETRY

MIAMI 2007

LAGO MAR RESORT HOTEL

Ft. Lauderdale, Florida

December 16, 2007

Paul H Frampton

UNC-CHAPEL HILL

OUTLINE

1. Historical context.
2. Neutrino mixing.
3. Tetrahedral $T = A_4$.
4. Binary tetrahedral $T^{'} = SL_2(F_3)$.
5. $T^{'}$ as flavor symmetry: A Model.
6. Formulas relating Quarks and Leptons.

Summary.

References:

P.D. Carr and P.H.F.
hep-ph/0701034 (2007).

P.H.F. and T.W. Kephart.
ArXiv:0706.1186
JHEP 09:110 (2007).

P.H.F. and S. Matsuzaki.
arXiv:0710.5928 (2007).

First use of $T^{'}$ as flavor group:

P.H.F. and T.W. Kephart,
Int. J. Mod. Phys. **10A,** 4869 (1995).
hep-ph/9409330.

1. Historical context

Progress in our knowledge of the three neutrino masses and mixings has been remarkable since SuperKamiokande found the first convincing evidence of non zero neutrino mass in 1998 by SuperKamiokande, and the Sudbury Neutrino Observatory (SNO) exceeded all expectations by abruptly solving the solar neutrino puzzle in 2001 thereby resolving the 35-year old conundrum set up by the persistent, and correct, experiments by Davis. It is probably fair to say that previously the majority of colleagues believed the data of Davis were explicable by suspected inaccuracies of the Standard Solar Model (SSM), but as we now know the SSM is a description of our Sun which is accurate to much better than a factor three, actually to within ten per cent.

It is also fair to say that our present knowledge of neutrino flavor is at least comparable to that of quark flavor despite the fact that the theory for quark flavor goes back to the 1963 article by Cabibbo (pre saged by a footnote in the 1960 paper by Gell-Mann and Lévy) and the paper by Gatto *et al.* in 1968. No complete understanding of the quark masses and mixings has subsequently emerged and the KM prediction of CP violation provides no insight into its magnitude.

2. Neutrino mixing.

We shall consider only three left-handed neutrinos at first*, so avoiding any encounter with the see-saw mechanism. The Majorana mass matrix $\mathcal{M}$ is a 3×3 unitary symmetric matrix and without CP violation has six real parameters. Let write the diagonal form as $\mathbf{M} = \mathrm{diag}(m_1, m_2, m_3)$, related to the flavor basis $\mathcal{M}$ by $\mathbf{M} = U^T \mathcal{M} U$ where U is orthogonal. It is the form of $\mathcal{M} = U \mathbf{M} U^T$ and U which are the targets of lepton flavor physics.

* Later we shall introduce three right-handed neutrinos and pursue the see-saw mechanism

One technique for analysis of $\mathcal{M}$ is to assume texture zeros in $\mathcal{M}$ and this gives rise to relationships between the mass eigenvalues m_i and the mixing angles θ_{ij}. For example, it has been shown that $\mathcal{M}$ cannot have as many as three texture zeros out of a possible six but can have two. A quite different interesting philosophy is that neutrino masses may arise from breaking of lorentz invariance. Clearly, a wide range of approaches is being aimed at the problem.

In the present study we focus on a symmetric texture for $\mathcal{M}$ with only four independent parameters, of the form

$$\mathcal{M} = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix} \tag{1}$$

The $\mathcal{M}$ can be reached from a diagonal $\mathbf{M}$ by the orthogonal transformation

$$U = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12}/\sqrt{2} & \cos\theta_{12}/\sqrt{2} & -1/\sqrt{2} \\ -\sin\theta_{12}/\sqrt{2} & \cos\theta_{12}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \tag{2}$$

where one commits to a relationship between θ_{12} and the four parameters in Eq.(1), namely

$$\tan 2\theta_{12} = 2\sqrt{2B(A-C-D)^{-1}} \tag{3}$$

Written in the standard PMNS form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} e^{i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13} e^{-1\delta} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4)$$

this ansatz requires that $\theta_{23} = \pi/4$ and $\theta_{13} = 0$, both of which are consistent with present data. These values of maximal θ_{23} and vanishing θ_{13} are presumably only approximate but departures therefrom, if they show up in future experiments, could be accommodated by higher order corrections.

To arrive at tribimaximal mixing one more parameter θ_{12} in Eq. (2) is assigned such that the entries of the second column are equal, *i.e.* $\sin\theta_{12} = \cos\theta_{12}/\sqrt{2}$ which implies that $\tan^2\theta_{12} = 1/2$. Experimentally θ_{12} is non-zero and over 5σ from a maximal $\pi/4$. The present value is $\tan^2\theta_{12} = 0.452^{+0.088}_{-0.070}$, so the tribimaximal value is within the allowed range. With this identification Eq.(2) becomes

$$U_{HPS} = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/6} & \sqrt{1/3} & -1/\sqrt{2} \\ -\sqrt{1/6} & \sqrt{1/3} & 1/\sqrt{2} \end{pmatrix} \qquad (5)$$

This ensures that the three mixing angles θ_{ij} are consistent with present data, although more accurate experiments may require corrections to U_{HPS}.

The data allow the normal hierarchy which occurs most often from models with $|m_3| \gg |m_{2,1}|$. In the normal hierarchy one expects $|m_3| \sim \sqrt{|\Delta_{23}|} \sim 0.05$ eV, $|m_2| \sim \sqrt{|\Delta_{12}|} \sim 0.009$ eV and $|m_1|$ essentially zero, as is the prediction for the eigenvalues in the FGY model. The data also allow for an inverted hierarchy with $|m_1| \sim |m_2| \gg |m_3|$. A third possible pattern is the degenerate case $|m_1| \sim |m_2| \sim |m_3| \gg |(m_3 - m_2)|$. The tribimaximal mixing, U_{HPS}, can accommodate all three of these neutrino mass patterns and so makes no prediction in that regard.

3. Tetrahedral $T = A_4$

The success of U_{HPS} tribimaximal neutrino mixing has precipitated many studies of its group theoretic basis and the tetrahedral group A_4 has emerged. The present analysis was prompted by earlier work in systematically studying *all* non-abelian finite groups of order $g \leq 31$ both for a quark flavor group and for orbifold compactification in string theory. Our question is whether or not A_4 is singled out from these as the neutrino flavor symmetry?

Character Table of $T \equiv A_4$

$\omega = \exp(2\pi i/3)$

	1_1	1_2	1_3	3
C_1	1	1	1	3
C_2	1	1	1	-1
C_3	1	ω	ω^2	0
C_4	1	ω^2	ω	0

Kronecker Products for Irreducible Representations of $T \equiv A_4$

	1_1	1_2	1_3	3
1_1	1_1	1_2	1_3	3
1_2	1_2	1_3	1_1	3
1_3	1_3	1_1	1_2	3
3	3	3	3	$1_1 + 1_2 + 1_3 + 3 + 3$

4. Binary tetrahedral $T^{'} = SL_2(F_3)$

The Kronecker products for irreducible representations for all the fourty-five non-abelian finite groups with order $g \leq 31$ have been explicitly tabulated with a presentation is adapted to a style aimed at model builders in theoretical physics rather than at mathematicians.

Study thereof shows that a promising flavor group is $\equiv SL_2(F_3)$. The Kronecker products are identical to those of $T \equiv A_4$ if the doublet representations are omitted and so the group $SL_2(F_3)$ can reproduce successes of A_4 model building. The use of $SL_2(F_3)$ as a flavor group first appeared in 1994.

$SL_2(F_3)$ has an advantage over T in extension to the quark sector because the doublets of $SL_2(F_3)$, absent in T, allow the implementation of a $(2+1)$ structure to the three quark families, thus permitting the third heavy family to be treated differently.

Character Table of $SL_2(F_3)$

$\omega = \exp(2\pi i/6)$

	1_1	1_2	1_3	2_1	2_2	2_3	3
C_1	1	1	1	2	2	2	3
C_2	1	1	1	-2	-2	-2	3
C_3	1	ω^2	ω^4	-1	ω^5	ω	0
C_4	1	ω^4	ω^2	-1	ω	ω^5	0
C_5	1	1	1	0	0	0	-1
C_6	1	ω^2	ω^4	-1	ω^2	ω^4	0
C_7	1	ω^4	ω^2	1	ω^4	ω^2	0

Kronecker Products for Irreducible Representations of $SL_2(F_3)$

	1_1	1_2	1_3	2_1	2_2	2_3	3
1_1	1_1	1_2	1_3	2_1	2_2	2_3	3
1_2	1_2	1_3	1_1	2_2	2_3	2_1	3
1_3	1_3	1_1	1_2	2_3	2_1	2_2	3
2_1	2_1	2_2	2_3	$1_1 + 3$	$1_2 + 3$	$1_3 + 3$	$2_1 + 2_2 + 2_3$
2_2	2_2	2_3	2_1	$1_2 + 3$	$1_3 + 3$	$1_1 + 3$	$2_1 + 2_2 + 2_3$
2_3	2_3	2_1	2_2	$1_3 + 3$	$1_1 + 3$	$1_2 + 3$	$2_1 + 2_2 + 2_3$
3	3	3	3	$2_1 + 2_2 + 2_3$	$2_1 + 2_2 + 2_3$	$2_1 + 2_2 + 2_3$	$1_1 + 1_2 + 1_3 + 3 + 3$

It is important to remark that $SL_2(F_3)$ does not contain T as a subgroup so our discussion about quark masses does not merely extend T, but replaces it.

The philosophy used for $SL_2(F_3)$ is reminiscent of much earlier work in in the 1990's where the dicyclic group Q_6 was used with doublets and singlets for the (1st, 2nd) and (3rd) families to transform as $(\mathbf{2} + \mathbf{1})$ respectively. On the other hand, Q_6 is not suited for tribimaximal neutrino mixing because like all dicyclic groups Q_{2n} it has no triplet representation. Recall that when the work on Q_6 was done, experiments had not established neutrino mixing for the reason mentioned in the Historical Context earlier.

5. $T^{'}$ as flavor symmetry: A Model.

The standard model (SM) of particle theory based on $SU(3) \times SU(2) \times U(1)$ suffers from a surfeit of parameters. With three quark-lepton families, non-zero neutrino masses and one Higgs doublet there are twenty-eight of them. Of particular interest for us here are the twenty parameters for masses and mixings of the quarks [ten: $\Theta_{ij}(1 \leq i, j \leq 3), \delta_{KM}, m_q^a(1 \leq a \leq 6)$], where δ_{KM} is the Kobayashi-Maskawa phase, and of the leptons [another ten: $\theta_{ij}(1 \leq i, j \leq 3), \delta_{PMNS}, m_l^b(1 \leq b \leq 6)$], where δ_{PMNS} is the Pontecorvo-Maki-Nakagawa-Sakata phase.

One approach to obtain non-trivial relationships between these parameters is to postulate a flavor (or horizontal) global symmetry G which commutes with the SM gauge group: $[SU(3) \times SU(2) \times U(1)]_{local} \times [G]_{global}$ and an especially appealing choice for G which we advocate involves $T^{'}$, the binary tetrahedral group. The present model employs $G = T^{'} \times Z_3 \times Z_3^{'} \times Z_3^{''}$.

Our global flavor symmetry will be based on the finite group $G \equiv T^{'} \times Z_3 \times Z_3^{'} \times Z_3^{''}$. We assign the left-handed quarks to the following representations of $T^{'} \times Z_3 \times Z_3^{'} \times Z_3^{''}$. Here the respective generators are denoted $\alpha = \exp(2\pi i/3)[Z_3]$, $\beta = \exp(2\pi i/3)[Z_3^{'}]$, $\gamma = \exp(2\pi i/3)[Z_3^{''}]$ and the representations are labelled by $(T^{'}, Z_3, Z_3^{'}, Z_3^{''})$:

For left-handed quarks:

$$\left.\begin{array}{l}\begin{pmatrix} t \\ b \end{pmatrix}_L \quad \mathcal{Q}_L \quad (1_1, 1, 1, 1) \\ \left.\begin{array}{l}\begin{pmatrix} c \\ s \end{pmatrix}_L \\ \begin{pmatrix} u \\ d \end{pmatrix}_L\end{array}\right\} Q_L \quad (2_1, 1, 1, 1),\end{array}\right. \tag{6}$$

and for the right-handed quarks

$$\begin{array}{c} t_R \quad (1_1, 1, 1, 1) \\ c_R \quad (1_3, 1, 1, 1) \\ u_R \quad (1_2, 1, 1, 1) \\ b_R \quad (1_2, 1, 1, 1) \\ \left.\begin{array}{c} s_R \\ d_R \end{array}\right\} \mathcal{S}_R \quad (2_3, \alpha, \beta, 1), \end{array} \tag{7}$$

and the leptons are assigned to

$$\left.\begin{array}{c}\left(\begin{array}{c}\nu_\tau\\ \tau^-\end{array}\right)_L\\ \left(\begin{array}{c}\nu_\mu\\ \mu^-\end{array}\right)_L\\ \left(\begin{array}{c}\nu_e\\ e^-\end{array}\right)_L\end{array}\right\} L_L \quad (3,\alpha^2,1,1) \quad \begin{array}{cc}\tau_R^- & (1_1,1,1,\gamma)\\ \mu_R^- & (1_2,1,1,\gamma)\\ e_R^- & (1_3,1,1,\gamma)\end{array} \quad \begin{array}{cc}N_R^{(1)} & (1_1,1,1,1)\\ N_R^{(2)} & (1_2,1,1,1)\\ N_R^{(3)} & (1_3,1,1,1).\end{array} \tag{8}$$

For the Higgs scalar sector we introduce the following fields:

$$H_{1_2}(1_2, 1, 1, 1), H_{1_3}(1_3, 1, 1, 1),$$
$$H_{2_1}(2_1, \alpha^2, \beta, 1), H_{2_2}(2_2, \alpha^2, \beta, 1), H_{2_3}(2_3, \alpha^2, \beta, 1),$$

$$H_3(3, \alpha, 1, 1), H_3^{'}(3, \alpha, 1, \gamma^2).$$

We shall henceforth refer to these simply as: H_{1_2}, H_{1_3}, H_{2_1}, H_{2_2}, H_{2_3}, H_3 and $H_3^{'}$.

The Yukawa couplings of the quark sector are

$$
\begin{aligned}
\mathcal{L}_Y &= Y_t(\mathcal{Q}_L t_R H_{1_1}) \\
&+Y_b(\mathcal{Q}_L b_R H_{1_3}) + Y_{tu}(\mathcal{Q}_L \{u_R\}_{1_2} H_{1_3}) + Y_{tc}(\mathcal{Q}_L \{c_R\}_{1_3} H_{1_2}) \\
&+Y_{D3}(\{Q_L\}_{2_1}\{\mathcal{S}_R\}_{2_3} H_3) \\
&+Y_{D2}(\{Q_L\}_{2_1} b_R H_{2_3}) + Y_U(\{Q_L\}_{2_1} t_R H_{2_1}) \\
&+Y_u(\{Q_L\}_{2_1}\{u_R\}_{1_2} H_{2_3}) + Y_c(\{Q_L\}_{2_1}\{c_R\}_{1_3} H_{2_2}) \\
&+\text{h.c.},
\end{aligned} \tag{9}
$$

where all Yukawa couplings are real $Y = Y^*$ with the exception of Y_{tu} which introduces the phase δ_{KM}.

The resultant up U and down D quark mass matrices are

$$U = \begin{bmatrix} Y_u v_{2_3}^{(2)} & Y_c v_{2_2}^{(2)} & Y_U v_{2_1}^{(2)} \\ -Y_u v_{2_3}^{(1)} & -Y_c v_{2_2}^{(1)} & -Y_U v_{2_1}^{(1)} \\ Y_{tu} v_{1_3} & Y_{tc} v_{1_2} & m_t \end{bmatrix}, \tag{10}$$

and

$$D = \begin{bmatrix} Y_{D3} v_3^{(1)} & \frac{1-i}{2} Y_{D3} v_3^{(2)} & Y_{D2} v_{2_3}^{(2)} \\ \frac{1-i}{2} Y_{D3} v_3^{(2)} & Y_{D3} v_3^{(3)} & -Y_{D2} v_{2_3}^{(1)} \\ 0 & 0 & m_b \end{bmatrix}, \tag{11}$$

where the superscripts refer to the component of the $T^{'}$ representation.

To arrive at the Cabibbo-Kobayashi-Maskawa matrix we select the vacuum alignmnent

$$v_3^{(1)} = 0\,, v_3^{(2)} \neq 0\,, v_3^{(3)} \neq 0, \qquad (12)$$

$$v_{2_3}^{(1)} \neq 0\,, v_{2_3}^{(2)} = 0, \qquad (13)$$

which lead to a relation for the Cabibbo angle Θ_C as follows.

The D matrix now has the texture

$$\mathcal{M} = \begin{bmatrix} 0 & \frac{1-i}{2} Y_{D3} v_3^{(2)} & 0 \\ \frac{1-i}{2} Y_{D3} v_3^{(2)} & Y_{D3} v_3^{(3)} & -Y_{D2} v_{2_3}^{(1)} \\ 0 & 0 & m_b \end{bmatrix}, \tag{14}$$

and defining

$$\epsilon = \frac{Y_{D2} v_{2_3}^{(1)}}{m_b}, \tag{15}$$

one can expand

$$\begin{aligned} Y_{D3} v_3^{(2)} &= \sqrt{2 m_d m_s} \left[1 + \frac{1}{4}\epsilon^2 + \cdots \right], \\ Y_{D3} v_3^{(3)} &= (m_s - m_d) \left[1 + \frac{1}{2}\epsilon^2 + \cdots \right]. \end{aligned} \tag{16}$$

Assuming $\epsilon \ll 1$ in Eq.(16) one finds

$$s_C = \sin\Theta_C = \sqrt{\frac{m_d}{m_s}}, \tag{17}$$

for the sine of the Cabibbo angle Θ_C, a formula which has been derived in equivalent form long ago. We can obtain other formulas which involve quark and lepton sectors. Here we illustrate the general method

The Yukawa couplings for the lepton sector are

$$\begin{aligned} \mathcal{L}_Y &= \frac{1}{2}M_1 N_R^{(1)} N_R^{(1)} + M_{23} N_R^{(2)} N_R^{(3)} \\ &+ \Big\{ Y_{N1}\left(L_L N_R^{(1)} H_3\right) + Y_{N2}\left(L_L N_R^{(2)} H_3 + L_L N_R^{(3)} H_3\right) \\ &+ Y_e\left(L_L e_R H_3'\right) + Y_\mu\left(L_L \mu_R H_3'\right) + Y_\tau\left(L_L \tau_R H_3'\right) + \text{h.c.}\Big\}. \end{aligned} \tag{18}$$

Of importance in this more unified approach to flavor symmetry is the role of the H_3 Higgs which acts as *messenger* between the quark and lepton sectors.

The mass matrix for the Majorana right-handed neutrinos is

$$M_N = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & 0 & M_{23} \\ 0 & M_{23} & 0 \end{pmatrix} \tag{19}$$

so that it has two degenerate eigenvalues.

After a see-saw mechanism in a basis where the charged lepton mass matrix is diagonalized, the light neutrino Majorana mass matrix takes the form:

$$\mathcal{M}_\nu = \frac{A}{3}[v_3^{(3)}]^2 \begin{bmatrix} (1+\sqrt{2}s_C)^2 & (\omega - \sqrt{2}s_C + 2s_C^2\omega^2)X & (\omega^2 - \sqrt{2}s_C + 2s_C^2\omega)X \\ (\omega^2 - \sqrt{2}s_C + 2s_C^2\omega)X & (1-\sqrt{2}s_C + 2s_C^2)X & \omega + 2\sqrt{2}s_C + 2s_C^2\omega^2 \\ (\omega - \sqrt{2}s_C + 2s_C^2\omega^2)X & \omega^2 + 2\sqrt{2}s_C + 2s_C^2\omega & (1-\sqrt{2}s_C + 2s_C^2)X \end{bmatrix}, \tag{20}$$

where $s_C = \sin\Theta_C = \sqrt{m_d/m_s}$ as in Eq.(17), $\omega = \exp(2\pi i/3)$ and

$$\begin{aligned} A &= \frac{Y_{N1}^2}{M_1} + \frac{2Y_{N2}^2}{M_{23}}, \\ X &= \frac{Y_{N1}^2 M_{23} - Y_{N2}^2 M_1}{Y_{N1}^2 M_{23} + 2Y_{N2}^2 M_1}. \end{aligned} \tag{21}$$

At this point, the predictive power of using $T^{'}$ symmetry becomes manifest. Our $\mathcal{M}_\nu$ depends on only two parameters, the Cabibbo angle s_C (which is measured) and X which depends on the spectrum of the $N_R^{(i)}$ (which is not).

The matrix $\mathcal{M}_\nu$ of Eq.(20) can be diagonalized by a PMNS matrix which generically depends on four parameters $\theta_{12}, \theta_{13}, \theta_{23}$ and δ_{PMNS}.

Therefore by elimination of X we will extract non-trivial relations between the parameters, the principal point of our discussion.

6.Formulas relating SM Parameters

The SM cries out for more unification between the quark and lepton sectors, a hope which once upon a time rested on the now somewhat disfavored idea of grand unification with its concomitant hierarchy problem. Using a flavor group provides an alternative approach to reduce the number of parameters without introducing a hierarchy between the weak scale and another scale. The hierarchy conundrum may thereby only be postponed but for purposes of making predictions testable by experiment one can be motivated by only flavor symmetry.

Returning to the counting of parameters in the PMNS mixing matrix, we can find from Eq.(20) that, given proximity to tribimaximal mixing, the PMNS phase (δ_{PMNS}) must satisfy

$$\tan(\delta_{PMNS})\Delta_{23} = \sqrt{3}\left(\frac{1-\sqrt{2}s_C}{1+\sqrt{2}s_C}\right), \qquad (22)$$

where $\Delta_{23} \equiv (\pi/4 - \theta_{23})$ is the deviation from maximal solar neutrino mixing and $s_C = \sin\Theta_C$ is the Cabibbo angle. According to the PDG 2006 data compilation, $\sqrt{2}\sin\Theta_C \simeq 0.32$ and $|\Delta_{23}| < 0.14$ so Eq.(22) requires $|\tan(\delta_{PMNS})| > 6.3$.

Provided θ_{13} is non-vanishing, this suggests CP violation for neutrinos is experimentally observable.

A second formula obtained by a linear expansion around tribimaximal neutrino mixing is for the ratio of neutrino mass eigenvalues

$$|m_3/m_2| = \left(\frac{1-\sqrt{2}s_C+2s_C^2}{(1+\sqrt{2}s_C)^2}\right). \qquad (23)$$

The right hand side of Eq.(23) is $\simeq 0.45$ which favors an inverted mass hierarchy.

Summary.

As has hopefully become evident, spontaneously broken global symmetries can reduce the number of independent variables characterizing masses and mixings of quarks and leptons and thereby lead to empirically testable relations. In particular flavor or horizontal symmetries involving $T^{'}$ merit further attention.

Thank you for your attention.