Functional Determinants and Renormalization with Lorentz Violation

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Overview of Talk

- Introduction to Functional Determinants, Yang-Mills, and Lorentz Violation
- One-loop bosonic results
- Inclusion of Fermions
- Ghost violation
- BRS Symmetry and Summary

Introduction to Functional Determinants

Basic Gaussian integral is of form

$$\int dx e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

Can be easily generalized to N dimensional matrix **A** in exponent

$$\int d^N x \ e^{-\vec{x}\cdot\mathbf{A}\cdot\vec{x}} = C\frac{1}{\sqrt{det(\mathbf{A})}}$$

where C is a constant.

This expression can be used to construct the functional determinant of a linear operator S

$$\frac{1}{\sqrt{det(S)}} = C \int \mathcal{D}\phi e^{-\langle\phi|S|\phi\rangle}$$

$\Rightarrow$ Integrals appear in path integral approach to field theory

Example application to scalar ϕ^4 field theory with action

$$S = \int d^4x \mathcal{L} = \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

leads to generating functional for connected Green functions

$$Z[j] = \int \mathcal{D}\phi \exp \left[iS + i \int d^4x \phi(x) j(x) \right]$$

(the measure is normalized such that $Z[0] = 1$)

Classical (extremal action)solution solves the equation of motion

$$(\partial_\mu \partial^\mu + m^2 - \frac{\lambda}{3!}\phi_{cl}^2)\phi_{cl} = j\phi_{cl}$$

Can expand field in terms of small fluctuations about classical action

$$\phi(x) \to \phi_{cl} + \phi$$

Generating functional in saddle point evaluation to O (ϕ^2)

$$Z[j] = e^{iS_{cl}[\phi_{cl}]} \int \mathcal{D}\phi exp\left(-i \int d^4x \left[\frac{1}{2}\phi \left\{\partial_\mu \partial^\mu + m^2 - \frac{\lambda}{2}\phi_{cl}^2\right\}\phi\right]\right)$$

yielding functional determinant of $\partial_\mu \partial^\mu + m^2 - \frac{\lambda}{2}\phi_{cl}^2$

Evaluation of determinant

$$Det\left[\partial_\mu\partial^\mu + m^2 - \frac{\lambda}{2}\phi_{cl}^2\right] = Det\left[P(1 + P^{-1}\Delta)\right]$$

with $P = \partial_\mu\partial^\mu + m^2$ denoting free field piece and $\Delta = -\frac{\lambda}{2}\phi_{cl}^2$
($DetP$ piece drops out from normalization)

Useful identity

$$\ln DetS = Tr \ln S$$

converts calculation to

$$Tr \ln(1 + P^{-1}\Delta) = Tr \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(P^{-1}\Delta)^n$$

$n = 1$ term

$$Tr(P^{-1}\Delta) = \frac{\lambda}{2}\left[\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 - m^2 + i\epsilon}\right]\int d^4x\phi_{cl}^2(x)$$

Regularize integral (dimensional regularization)

$$\int \frac{d^dp}{(2\pi)^d}\left(\frac{1}{p^2 - m^2 + i\epsilon}\right) = -\frac{i\Gamma(1 - d/2)}{(4\pi)^{d/2}}\frac{1}{(m^2)^{1-d/2}} \rightarrow \frac{im^2}{8\pi^2\epsilon}$$

has simple pole as $d \rightarrow 4$ ($d = 4 - \epsilon$) and

$$Tr(P^{-1}\Delta) = \frac{i\lambda m^2}{16\pi^2\epsilon}$$

Comparing this term with the classical action term

$$\ln Z = iS_{cl} - \frac{1}{2}\ln Det(P^{-1}\Delta) = im^2(1 - \frac{\lambda}{32\pi^2\epsilon})\int d^4x\phi_{cl}^2$$

Makes it immediately obvious that the pole can be absorbed into the mass and gives the appropriate multiplicative renormalization constant (valid to one-loop)

$$\Rightarrow \boxed{m_r = 1 - \tfrac{\lambda}{32\pi^2\epsilon}m}$$

This procedure may be repeated for the next order ($n = 2$) expansion in the $Tr\ln(1 + P^{-1}\Delta)$ term

$$\Rightarrow \boxed{\lambda_r = \lambda\left(1 + \tfrac{6\lambda}{32\pi^2\epsilon}\right)}$$

Introduction to Yang-Mills

Constructed using Lie-Algebra valued vector potential A^μ to form covariant derivative $D^\mu = \partial^\mu + igA^\mu$ and gauge covariant curvature

$$F^{\mu\nu} = -\frac{i}{g}[D^\mu, D^\nu]$$

Pure Yang-Mills action given by

$$S_{YM} = -\frac{1}{2} tr \int d^4 x F_{\mu\nu} F^{\mu\nu}$$

Impose gauge fixing using Faddeev-Popov procedure

$\rightarrow$ introduce gauge functional $\mathcal{F}[A]$ and ghost fields c

$$S_{FP} = -\frac{1}{2} tr \int d^4x \left[F_{\mu\nu} F^{\mu\nu} + 2\lambda \mathcal{F}[A]^2 - \overline{c} \mathcal{M} c \right]$$

where $\mathcal{M}$ is variation of $\mathcal{F}$ w.r.t. gauge transformation

$$\mathcal{M} = \frac{\delta \mathcal{F}[A]}{\delta A^\mu} D^\mu$$

Example: $\mathcal{F} = \partial_\mu A^\mu \rightarrow \mathcal{M} = \partial_\mu D^\mu$

Note that it is also possible to choose a Lorentz-violating gauge

$$\mathcal{F} = (\partial_\mu + C_{\mu\nu} \partial_\nu) A^\mu \rightarrow \mathcal{M} = (\partial_\mu + C_{\mu\nu} \partial_\nu) D^\mu$$

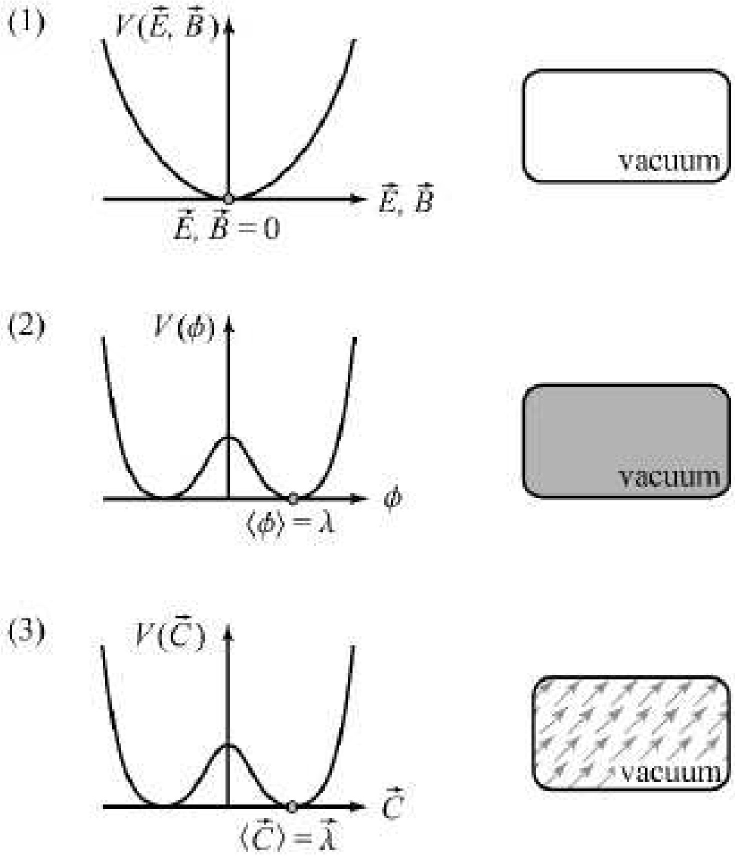
Motivation and Implementation of Lorentz Violation

- Generic theories underlying standard model ”naturally” allow for Lorentz-breaking effects: Original conception in string theory
  - Kostelecky and Samuel 1989

- Incorporate into standard model using effective field theory to generate Standard Model Extension (SME)
  - DC and Kostelecky 1997

$\Rightarrow$ advantage of model independent framework

One way of implementation: Spontaneous Symmetry Breaking



- Lehnert hep-ph/0611177

Some experimental tests of CPT/Lorentz violation

(CPT '07' conference in August, Indiana)

- Neutrino Oscillations
- Neutral-Meson Oscillations
- Clock-comparison tests
- Spin-polarized torsion pendulum
- H and $\overline{H}$ spectroscopy

- QED tests in Penning traps
- Muon propagation properties
- Measurements of cosmological birefringence
- Microwave and optical cavity measurements
- Observation of baryon asymmetry
- Tests of post-Newtonian gravity

Including Lorentz Violation into Yang-Mills

If want to maintain gauge invariance and power-counting renormalizability, can include

$$\mathcal{L}_{CPT-even} = -\frac{1}{2}(k_F)_{\mu\nu\alpha\beta} tr F^{\mu\nu} F^{\alpha\beta}$$

$$\mathcal{L}_{CPT-odd} = -\frac{1}{2}(k_{AF})^{\kappa} \epsilon_{\kappa\lambda\mu\nu} tr(A^{\lambda} F^{\mu\nu} - \tfrac{2}{3} ig A^{\lambda} A^{\mu} A^{\nu})$$

In this expression, $(k_F)^{\mu\nu\alpha\beta}$ and $(k_{AF})^{\kappa}$ are constant tensor background fields that break Lorentz invariance

Doing this for all fields → Standard Model Extension (SME)

Some related work on renormalization in SME

- One Loop Renormalization of Lorentz-violating QED
  Kostelecky, Lane, and Pickering, PRD 2002

- Lorentz-violating QED renormalization in curved space
  Barredo-Peixoto and Shapiro, PLB 2006

- Lorentz violation and Faddeev-Popov Ghosts
  Altschul, PRD 2006

- New Bounds on Isotropic Lorentz Violation
  Carone, Sher, and Vanderhaeghen, PRD 2006

- One-Loop Renorm of Lorentz-violating Pure Yang Mills
  D. C., McDonald, PRD 2007

One-Loop Effective Action

$$\exp i\Gamma[\Psi] = \int \mathcal{D}\Psi e^{i\int \mathcal{L}(\Psi, \partial^\mu \Psi)}$$

where Ψ represents all dynamic fields

Expand gauge fields about classical background field $\underline{A}^\mu$

$$A^\mu = \underline{A}^\mu + \mathcal{A}^\mu$$

Leads to expression for curvature

$$F^{\mu\nu} = \underline{F}^{\mu\nu} + \underline{D}^{[\mu}\mathcal{A}^{\nu]} + ig[\mathcal{A}^\mu, \mathcal{A}^\nu]$$

where

$$\underline{D}^\mu = \partial^\mu + ig\underline{A}^\mu$$

is the covariant derivative with respect to background field

CPT-even gauge field terms

Saddle point evaluation requires quadratic (in $\mathcal{A}$) contribution

$$\begin{aligned}\mathcal{L}_{\mathcal{A}}^{quad} = \quad & -\frac{1}{2g^2} tr \mathcal{A}^{\mu} \left[-g_{\mu\nu} \underline{D}^2 - 2i \underline{F}_{\mu\nu} \right. \\ & \left. -i (k_F)_{\alpha\beta\mu\nu} \underline{F}^{\alpha\beta} - 2 (k_F)_{\mu\alpha\nu\beta} \underline{D}^{\alpha} \underline{D}^{\beta} \right] \mathcal{A}^{\nu} \ .\end{aligned}$$

The trace is extended to cover Lorentz indices using notation

$$\begin{aligned}(\tau_{\alpha\beta})_{\mu\nu} &= i(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu}) \\ (k^{I}_{F\alpha\beta})_{\mu\nu} &= (k_F)_{\alpha\beta\mu\nu} \\ (k^{II}_{F\alpha\beta})_{\mu\nu} &= (k_F)_{\mu\alpha\nu\beta}\end{aligned}$$

then

$$\mathcal{L}_{\mathcal{A}}^{quad} = -\frac{1}{2g^2} tr \mathcal{A} (\Delta_A) \mathcal{A}$$

Contributes $Det\Delta_A$ to effective action

$$\Delta_A = P_A + \Delta_A^{(1)} + \Delta_A^{(2)} + \Delta_A^{(F)}$$

$$\begin{aligned} P_A &= -(g_{\alpha\beta} + 2k^{II}_{F\alpha\beta})\partial^\alpha\partial^\beta \\ \Delta_A^{(1)} &= -i(g_{\alpha\beta} + 2k^{II}_{F\alpha\beta})(\partial^\alpha \underline{A}^\beta + \underline{A}^\alpha\partial^\beta) \\ \Delta_A^{(2)} &= (g_{\alpha\beta} + 2k^{I}_{F\alpha\beta})(\underline{A}^\alpha\underline{A}^\beta) \\ \Delta_A^{(F)} &= -[\tau_{\alpha\beta} + ik^{I}_{F\alpha\beta}]\underline{F}^{\alpha\beta} \end{aligned}$$

The first term P_A is independent of $\underline{A}$ and factors out

$$\ln Det({P_A}^{-1}\Delta_A) = \ln Det\left[1 + P_A^{-1}(\Delta_A^{(1)} + \Delta_A^{(2)} + \Delta_A^{(F)})\right].$$

Using the relationship

$$\ln Det S = Tr \ln S$$

and expanding the ln to second-order gives

$$\ln Det(P_A^{-1}\Delta_A) = \frac{i}{(4\pi)^2} C_2(G)\Gamma(2-\tfrac{d}{2})$$

$$\otimes \int \frac{d^4k}{(2\pi)^4} \Big[\left(\frac{7}{3}\right) k_{F\mu\lambda\nu}{}^{\lambda} Q^{\mu\nu} - (12) k_{F\mu\alpha\nu\beta}(k^\alpha k^\beta \underline{A}^\mu \underline{A}^\nu)$$

where

$$Q^{\mu\nu} = (k^\mu k^\nu \underline{A}^2 - 2k^\mu \underline{A}^\nu k \cdot \underline{A} + k^2 \underline{A}^\mu \underline{A}^\nu)$$

⇒ New result: Gives contribution of trace k_F terms neglected previousl

Addition of the corresponding term in classical effective action

$$e^{i\Gamma[\underline{A}]} = e^{iS_{cl}[\underline{A}]}(Det\Delta_A)^{-1/2}$$

yields one-loop divergent contribution to (trace-free) k_F coupling

$$\mathcal{L}_{eff} \supset -\frac{1}{4g^2}\left(1 - \frac{6g^2}{(4\pi)^2}\Gamma(2-\tfrac{d}{2})\right)(k_F)_{\mu\alpha\nu\beta}\underline{F}^{\mu\nu}\underline{F}^{\alpha\beta}$$

Allows immediate identification of renormalization parameters

$$\begin{aligned} g_b &= Z_g g_r \\ (k_F)_b &= Z_{k_F}(k_F)_r \end{aligned}$$

$$Z_g = 1 - (11/6)\frac{g^2 C_2(G)}{(4\pi)^2}\Gamma(2-\tfrac{d}{2}). \qquad (1)$$

$$\boxed{Z_{k_F} = 1 + (7/3)\frac{g^2 C_2(G)}{(4\pi)^2}\Gamma(2-\tfrac{d}{2})} \qquad (2)$$

Trace k_F terms take the form

$$k_F^{\mu\nu\alpha\beta} = \Lambda^{\mu\alpha} g^{\nu\beta} - \Lambda^{\nu\alpha} g^{\mu\beta} - \Lambda^{\mu\beta} g^{\nu\alpha} + \Lambda^{\nu\beta} g^{\mu\alpha}$$

where $\Lambda^{\mu\nu}$ is symmetric matrix

Comparison to classical term gives

$$\mathcal{L}_{eff} \supset -\frac{1}{g^2}\left(1 - \frac{11}{3}\frac{C_2(G)g^2}{4\pi^2}\Gamma(2-\tfrac{d}{2})\right) \Lambda^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} Q^{\mu\nu}$$

where again

$$Q^{\mu\nu} = (k^\mu k^\nu \underline{A}^2 - 2k^\mu \underline{A}^\nu k \cdot \underline{A} + k^2 \underline{A}^\mu \underline{A}^\nu)$$

Divergence already absorbed using $g_b = Z_g g_r$

$\Rightarrow$ $\boxed{\text{No renormalization of } \Lambda^{\mu\nu} \text{ required (without fermions)}}$

CPT-odd gauge field terms

form of $Det\Delta_A$ contribution to effective action

$$\Delta_A = P_A + \Delta_A^{(1)} + \Delta_A^{(2)} + \Delta_A^{(F)}$$

$$\begin{aligned} P_A &= -(g_{\alpha\beta} + ((k_{AF})^\kappa \epsilon_{\kappa\beta}))\partial^\alpha \partial^\beta \\ \Delta_A^{(1)} &= -ig_{\alpha\beta}(\partial^\alpha \underline{A}^\beta + \underline{A}^\alpha \partial^\beta) - i(k_{AF})^\alpha \epsilon_{\alpha\beta}\underline{A}^\beta \\ \Delta_A^{(2)} &= g_{\alpha\beta}\underline{A}^\alpha \underline{A}^\beta \\ \Delta_A^{(F)} &= -\tau_{\alpha\beta}\underline{F}^{\alpha\beta} \end{aligned}$$

where $\epsilon_{\alpha\beta}$ is matrix notation for $(\epsilon_{\alpha\beta})_{\mu\nu} = \epsilon_{\alpha\beta\mu\nu}$

$\Rightarrow$ k_{AF} contributions to functional determinant cancel!

$$\boxed{Z_g^2 = Z_{k_{AF}}}$$

Inclusion of Fermions

Fermion piece of lagrangian is

$$\mathcal{L}_\psi = \bar{\psi}(i\Gamma^\mu \underline{D}_\mu - M)\psi$$

where $\Gamma^\nu = \gamma^\nu + \Gamma_1^\nu$, $M = m + M_1$,
with Lorentz-violating parameters

$$\begin{aligned} \Gamma_1^\nu &= c_{\nu\mu}\gamma^\mu + d_{\mu\nu}\gamma_5\gamma_\mu + e_\nu + if_\nu\gamma_5 + \tfrac{1}{2}g^{\lambda\mu\nu}\sigma_{\lambda\mu} \\ M_1 &= a_\mu\gamma^\mu + b_\mu\gamma_5\gamma^\mu + \tfrac{1}{2}H_{\mu\nu}\sigma^{\mu\nu}. \end{aligned}$$

Integral over ψ gives functional determinant

Convenient to square operator for calculation
(no divergent contribution from M)

$$-(\Gamma^\mu D_\mu)^2 = -P_\psi(1 - P_\psi^{-1}(\Delta_\psi^{(1)} + \Delta_\psi^{(2)} + \Delta_\psi^{(F)}))$$

where

$$\begin{aligned}
P_\psi &= (g^{\alpha\beta} + \{\gamma^\alpha, \Gamma_1^\beta\})\partial_\alpha\partial_\beta \\
\Delta_\psi^{(1)} &= -i(g^{\alpha\beta} + \{\gamma^\alpha, \Gamma_1^\beta\})(\partial_\alpha \underline{A}_\beta + \underline{A}_\alpha \partial_\beta) \\
\Delta_\psi^{(2)} &= (g^{\alpha\beta} + \{\gamma^\alpha, \Gamma_1^\beta\})\underline{A}_\alpha \underline{A}_\beta \\
\Delta_\psi^{(F)} &= -(S^{\alpha\beta} + \tfrac{1}{2}[\gamma^\alpha, \Gamma_1^\beta])\underline{F}_{\alpha\beta},
\end{aligned}$$

$\Rightarrow$ Only $c^{\mu\nu}$ has correct symmetry properties to contribute

Calculation of determinant yields fermion contribution

$$\ln Det(P_\psi^{-1}\Delta_\psi) = -\frac{i}{3}\frac{C(r)}{(4\pi)^2}\Gamma(2-\frac{d}{2})c_{\mu\nu}\int\frac{d^4k}{(2\pi)^4}Q^{\mu\nu} \qquad (3)$$

where

$$Q^{\mu\nu} = (k^\mu k^\nu \underline{A}^2 - 2k^\mu \underline{A}^\nu k\cdot\underline{A} + k^2\underline{A}^\mu\underline{A}^\nu)$$

defining

$$\Lambda_b^{\mu\nu} = (Z_\Lambda)^{\mu\nu}_{\alpha\beta}\Lambda_r^{\alpha\beta}$$

gives the following renormalization condition

$$(Z_\Lambda)^{\mu\nu}_{\alpha\beta}\left[\Lambda_b^{\alpha\beta} + \frac{1}{6}\frac{g^2}{(4\pi)^2}\Gamma(2-\frac{d}{2})\left(C(r)c^{\alpha\beta} + C_2(G)C^{\alpha\beta}\right)\right] = Z_S\Lambda_b^{\mu\nu} ,$$

and the scalar coefficient Z_S (due to Z_g) is given by

$$Z_S = \left(1 + \frac{4}{3}\frac{g^2}{(4\pi)^2}\Gamma(2-\frac{d}{2})n_f C(r)\right)$$

Ghost Violation

Can include extra Lorentz-breaking terms directly in ghost sector

$$\mathcal{L}_G = -\overline{c}(-\underline{D}^\mu \underline{D}_\mu - C_{\mu\nu}\underline{D}^\mu \underline{D}^\nu)c$$

yields additional functional determinant contribution

$$\mathrm{logdet}(P_c^{-1}\Delta_c) = -\frac{i}{6}\frac{C_2(G)}{(4\pi)^2}\Gamma(2-\tfrac{d}{2})C_{\mu\nu}\int\frac{d^4k}{(2\pi)^4}Q^{\mu\nu}. \qquad (4)$$

where again

$$Q^{\mu\nu} = (k^\mu k^\nu \underline{A}^2 - 2k^\mu \underline{A}^\nu k\cdot\underline{A} + k^2 \underline{A}^\mu \underline{A}^\nu)$$

⇒ can be renormalized into Z_Λ (trace piece of k_F)

Becchi-Rouet-Stora (BRS) Invariance

Action for gauge particles (A) and ghosts (c) is

$$S = \int d^4x \mathcal{L} = \int d^4x \left(\mathcal{L}_0[F] - \frac{\lambda}{2}\mathcal{F}^2 - \bar{c}\mathcal{M}c \right)$$

where $\mathcal{F}[A]$ is gauge fixing functional and

$$\mathcal{M} = \frac{\delta \mathcal{F}}{\delta A^\mu} D^\mu$$

gives the ghost contribution

Action is invariant under generalized gauge transformation (BRS) *provided* $\mathcal{L}_0$ is gauge invariant functional of F, and no ghost LV

$\Rightarrow$ true for Lorentz-breaking terms k_F and k_{AF}

BRS transformation explicitly given by

$$\begin{aligned} \delta A^\mu &= D^\mu c \delta\zeta \\ \delta\overline{c} &= \lambda\mathcal{F}[A]\delta\zeta \\ \delta c_a &= -\frac{1}{2} f_{abc} c_b c_c \delta\zeta \end{aligned}$$

where $\delta\zeta$ is anticommuting parameter

(BRS used for all-orders renormalization in conventional case...)

Summary

- Functional determinant technique provides convenient way to calculate loop corrections in Standard Model Extension(SME)
- BRS transformation remains symmetry even though Lorentz-violation is present: This should be beneficial in proving all-orders renormalizability of theory
- Now have a realistic model for QCD with Lorentz violation at one loop involving both gluonic and quark contributions to gluon propagator