MIAMI 2007 CONFERENCE SCALAR-TENSOR INTERACTION IN FRW UNIVERSES WITH BAROTROPIC FLUIDS

P. Ag. Chauvet*
Departamento de Física ,
Universidad Autónoma Metropolitana–Iztapalapa
P. O. Box. 55–534, México D. F.
C. P. 09340 MEXICO
Fax 52-5-724 4611

January 21, 2008

The field equations for perfect fluids in Friedman–Lemaître Brans–Dicke models include two time dependent variables. However, its governing field equations can be cast as non–linear equations for a single variable, similar to the corresponding Einstein's model equations, which in these cases comprise the scale factor to describe their evolution as its only time unknown. So in a sence, this scalar–tensor theory has just one degree of freedom.

Barotropic fluids are perfect fluids distinguished through a constant called the barotropic factor n, which plays a role in the construction of the aforementioned key variable that, by itself, describes the cosmic unfolding of these fluids. The functions are scalar field and scale factor combinations. Moreover, the new equations permit one to obtain its analytical solutions in a more direct way its rather surprising physical implications that are also made apparent. Indeed, the curved space models for these fluid are found to have an unexpected behavior that may discarded them as vailable models capable scalar field with the scale factorof representing the universethat could imply they are not on an equal footing with the flat space.

The line element for the FRW metric in (t, r, θ, φ) coordinates is

$$ds^2 = -dt^2 + a(t)^2[(1 - kr^2)^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi)]$$

*e-mail:pcha@xanum.uam.mx

where a is the scale factor, and $k = +1$, 0 or -1 for the positive, vanishing or negative space curvature, respectively.

Without regard to the magnitud of ω significant features of BD have passed unoticed. That can hardly be inferred from the original form in which its equations were written: The most salient obstacle is one has to solve a set of non linear, coupled differential equations for two time dependent variables [?]. However these equations truly represent equations for only a single variable through which the cosmic evolution of every FRW space curvature is displayed. Weinberg [?] seems to be the first one to produce an equation for what he called a single auxiliary function that combines the scalar field with the scale factor to land the expansion for the BD flat space dust models only. A single function also renders the isotropic expansion of other barotropic fluids models, including those with a vacuous stress–energy tensor, as for some anisotropic models [?] as well. The velocity of light is $c = 1$.

1 The JBD Scalar–Tensor Lagrangian and its Field Equations for Barotropic Fluids

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= 8\pi \quad \frac{T_{\mu\nu}}{\phi} + \frac{1}{\phi}[\phi_{;\mu\nu} - g_{\mu\nu}\phi_{;\alpha}\phi^{;\alpha}] \\
&+ \quad \frac{\omega}{\phi}[\phi_{;\mu}\phi_{;\nu} - \frac{1}{2} g_{\mu\nu}\phi_{;\alpha}\phi^{;\alpha}] \quad , \qquad (1)
\end{aligned}
$$

and

$$
\phi_{;\mu}{}^{;\mu} = \left(\frac{8\pi}{3+2\omega}\right) T \quad . \qquad (2)
$$

Here, the energy–momentum tensor for a perfect fluid is

$$
T^{\mu\nu} = (\rho + p) u^{\mu} u^{\nu} \; + \; p g^{\mu\nu}
$$

u^{β} is its four velocity, and $T \equiv T_{\mu}{}^{\mu}$ its trace. The equation

$$
T^{\nu}_{\mu;\nu} = 0 \qquad (3)
$$

establishes the matter conservation law.

For perfect fluids the FRW, BD cosmology the field equations comprise two unknown variables $a = a(t)$ and $\phi = \phi(t)$:

$$
\frac{\ddot{\phi}}{\phi} + 3\frac{\dot{a}\dot{\phi}}{a\phi} = \frac{8\pi}{3+2\omega} \frac{(\rho - 3p)}{\phi} \quad , \qquad (4)
$$

is the ϕ field equation, together with

$$
\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{\dot{a}\dot{\phi}}{a\phi} + 2\frac{k}{a^2} = \frac{8\pi}{3+2\omega} \frac{[(1+\omega)\rho - \omega p]}{\phi} \quad , \qquad (5)
$$

and

$$3\frac{\ddot{a}}{a} + \omega\frac{\dot{\phi}^2}{\phi^2} + \frac{\ddot{\phi}}{\phi} = \frac{-8\pi}{3+2\omega}\,\frac{[(2+\omega)\rho + 3(1+\omega)p]}{\phi} \qquad , \tag{6}$$

Barotropic fluids imply the relation $p = n\rho$ with $-1 < n < 1$, where p is the pressure, and ρ its energy density. With the notation $d(\ )/dt \equiv (\ )^{\dot{}}$, the conservation equation for barotropic fluids

$$\dot{\rho} = -3(\rho + p)(\ln a)^{\dot{}}$$

yields

$$\rho = M_n a^{-3(1+n)} , \qquad M_n = \text{const.} \tag{7}$$

If instead of t the "time parameter" η, defined by $dt = a^{3n}d\eta$, is used with $()' = \partial\eta$, and the notation $q \equiv \phi a^{3(1-n)}$, the ϕ field equation Eq.(4) integrates as

$$\begin{aligned} d(\dot{\phi}a^3) &= (1-3n)m_n a^{3n}dt = (1-3n)m_n d\eta \\ \dot{\phi}a^3 &= (1-3n)m_n\eta + \eta_0 \quad , \quad m_n = \frac{8\pi M_n}{3+2\omega} \end{aligned} \tag{8}$$

and η_0 is an integration constant. Therefore, since $\dot{a}/a = a^{-3n}(a'/a)$, and $\dot{\phi}/\phi = a^{-3n}(\phi'/\phi)$

$$3(1-n)\frac{a'}{a} = \frac{q'}{q} - \frac{(1-3n)m_n\eta + \eta_0}{q} \quad , \quad m_n = \frac{8\pi M_n}{3+2\omega} \tag{9}$$

where

$$\frac{\phi'}{\phi} = \frac{(1-3n)m_n\eta + \eta_0}{q} = \frac{\xi}{q} \quad , \tag{10}$$

η_0 is an integration constant.

So that, together with the ϕ field equation any barotropic fluid model, excluding radiation (the $n = 1/3$ case is excluded from the above, and treated as a separate case) is described by the following two equations

$$\begin{aligned} (1-3n)q'' \quad - \quad & [2(2-3n) + 3(1-n)^2\omega]\xi' = -6(1-n)(1-3n)k \times \\ & \exp\left\{\frac{1}{3(1-n)}\int\left\{(1+3n)\frac{q'}{q} + 2(1-3n)\frac{\xi}{q}\right\}d\eta\right\} \end{aligned} \tag{11}$$

since

$$a = \exp\left\{\frac{1}{3(1-n)}\int\left(\frac{q'}{q} - \frac{\xi}{q}\right)d\eta\right\} \qquad ,$$

and where $k = 0$, $+1$, or -1.

$$\begin{aligned} & \frac{q''}{q} - \frac{2}{3(1-n)}\frac{q'^2}{q^2} + \frac{[2(2-3n) + 3(1-n)^2\omega]}{3(1-n)}\frac{\xi^2}{q^2} \\ - \quad & \frac{2(1-3n)}{3(1-n)}\frac{\xi q'}{q^2} + \frac{[2 + (1+3n)(1-n)\omega]}{(1-3n)}\frac{\xi'}{q} = 0\,. \end{aligned} \tag{12}$$

For comparison purposes however, multiplying by $(1+3n)$ it displays as

$$\left\{(1+3n)\frac{q'}{q}+2(1-3n)\frac{\xi}{q}\right\}'+\frac{1-3n}{3(1-n)(1+3n)}\left\{(1+3n)\frac{q'}{q}+2(1-3n)\frac{\xi}{q}\right\}^2$$
$$+\frac{(1-n)[18n+(1+3n)^2\omega]}{(1-3n)(1+3n)}\left\{(1+3n)\frac{\xi'}{q}+(1-3n)\frac{\xi^2}{q^2}\right\}=0 \quad . \qquad (13)$$

From Eq.(11), the third q derivative is

$$(1-3n)\frac{d^3q}{d\eta^3}=\frac{1}{3(1-n)}\left\{(1-3n)q''-[2(2-3n)+3(1-n)^2\omega]\xi'\right\}\times$$
$$\left\{(1+3n)\frac{q'}{q}+2(1-3n)\frac{\xi}{q}\right\} \qquad (14)$$

while its fourth derivative is

$$\begin{aligned}(1-3n)\frac{d^4q}{d\eta^4} &= \left\{(1-3n)q''-[2(2-3n)+3(1-n)^2\omega]\xi'\right\}\times\frac{1+3n}{3(1-n)}\left\{\frac{q''}{q}\right.\\ -\frac{2(1-3n)}{3(1-n)}\frac{q'^2}{q^2} &+ \frac{4(1-3n)^2}{3(1-n)(1+3n)}\frac{\xi^2}{q^2}-\frac{2(1-3n)(1-9n)}{3(1-n)(1+3n)}\frac{\xi q'}{q^2}\\ \left.+\frac{2(1-3n)}{(1+3n)}\frac{\xi'}{q}\right\} & \quad . \end{aligned} \qquad (15)$$

2 Linear Fractional Transformations

Eq.(12) laid out as

$$\begin{aligned}3(1-n)qq''-2q'^2-2(1-3n)\xi q' &+ [2(2-3n)+3(1-n)^2\omega]\xi^2\\ +\frac{3(1-n)}{(1-3n)}[2+(1+3n)(1-n)\omega]q\xi' &= 0\end{aligned} \qquad (16)$$

with $d\xi=(1-3n)m_n d\eta$, converts into

$$\begin{aligned}&3(1-n)(1-3n)^2{m_n}^2qq_{\xi\xi}-2(1-3n)^2{m_n}^2{q_\xi}^2+\\ -\;&2(1-3n)^2m_n\xi q_\xi+[2(2-3n)+3(1-n)^2\omega]\xi^2\\ +\;&3(1-n)[2+(1+3n)(1-n)\omega]m_nq=0\,.\end{aligned} \qquad (17)$$

The notation used in above stands for $q_\xi\equiv dq/d\xi$, and $q_{\xi\xi}\equiv d^2q/d\xi^2$. With the following definitions $\zeta=q\xi^{-2}=qe^{-2\tau}$, that is, $\tau=\ln\xi$ means

$$d/d\xi=e^{-\tau}d/d\tau,$$

and

$$d^2/d\xi^2=e^{-2\tau}(d^2/d\tau^2-d/d\tau)$$

which implies

$$\begin{aligned} d^2(\zeta e^{2\tau})/d\tau^2 &= d(\zeta_\tau e^{2\tau} + 2\zeta e^{2\tau})/d\tau \\ &= (\zeta_{\tau\tau} + 4\zeta_\tau + 4\zeta)e^{2\tau} . \end{aligned} \tag{18}$$

Therefore, with

$$q_\tau = (\zeta_\tau + 2\zeta)e^\tau$$

and

$$q_{\tau\tau} = (\zeta_{\tau\tau} + 3\zeta_\tau + 2\zeta)e^{2\tau}$$

substituted into Eq.(12) it transforms to

$$\begin{aligned} & 3(1-n)(1-3n)^2{m_n}^2(\zeta\zeta_{\tau\tau} + 3\zeta\zeta_\tau + 2\zeta^2) \\ - & \; 2(1-3n)^2{m_n}^2({\zeta_\tau}^2 + 4\zeta\zeta_\tau + 4\zeta^2) \\ - & \; 2(1-3n)^2 m_n(\zeta_\tau + 2\zeta) \\ + & \; 3(1-n)[2 + (1+3n)(1-n)\omega]m_n\zeta \\ + & \; [2(2-3n) + 3(1-n)^2\omega] \; = \\ & 3(1-n)(1-3n)^2{m_n}^2\zeta\zeta_{\tau\tau} - 2(1-3n)^2{m_n}^2{\zeta_\tau}^2 \\ + & \; (1-3n)^2[(1-9n)m_n\zeta - 2]m_n\zeta_\tau \\ - & \; 2(1+3n)(1-3n)^2{m_n}^2\zeta^2 \\ + & \; [2(1+9n-18n^2) + 3(1+3n)(1-n)^2\omega]m_n\zeta \\ + & \; [2(2-3n) + 3(1-n)^2\omega] = 0 , \end{aligned} \tag{19}$$

and with $m_n\zeta \equiv y$ obtains

$$\begin{aligned} & 3(1-n)(1-3n)^2 y y_{\tau\tau} - 2(1-3n)^2 {y_\tau}^2 \\ + & \; (1-3n)^2[(1-9n)y - 2]y_\tau - 2(1+3n)(1-3n)^2 y^2 \\ + & \; [2(1+9n-18n^2) + 3(1+3n)(1-n)^2\omega]y \\ + & \; [2(2-3n) + 3(1-n)^2\omega] \; = 0 , \end{aligned} \tag{20}$$

which, for $n = 0$ is

$$3yy_{\tau\tau} - 2{y_\tau}^2 + (y-2)y_\tau - 2y^2 + (2+3\omega)y + (4+3\omega) = 0 . \tag{21}$$

With the following substitutions, $y_\tau \equiv x$ and $dx/dy \equiv x_y$, Eq.(20) is turned into the next, a first order equation,

$$\begin{aligned} yxx_y \; - & \; \frac{2}{3(1-n)}x^2 + \frac{[(1-9n)y-2]}{3(1-n)}x \\ + & \; \frac{[2(2-3n) + 3(1-n)^2\omega]}{3(1-n)(1-3n)^2} \\ + & \; \frac{[4 - 2(1-3n)(1-6n) + 3(1+3n)(1-n)^2\omega]}{3(1-n)(1-3n)^2}y \\ - & \; \frac{2(1+3n)}{3(1-n)}y^2 \; = 0 . \end{aligned} \tag{22}$$

Eliminating from it the non linear term x^2 through the substitution $x \equiv y^{2/3(1-n)}z$, Eq.(22) simplifies furthermore

$$zz_y + z\frac{[(1-9n)y-2]}{3(1-n)}y^{\frac{-5+3n}{3-3n}} - \frac{[(1+3n)y+1]}{3(1-n)(1-3n)^2} \quad \times$$
$$\left\{2(1-3n)^2y - 2(2-3n) - 3(1-n)^2\omega\right\} y^{\frac{-7+3n}{3-3n}} \qquad = 0 \qquad (23)$$

this is an Abel equation of the Second Kind, and through

$$s = \frac{1}{3(1-n)}\int [(1-9n)y-2]y^{\frac{-5+3n}{3-3n}}\, dy$$

it turns into

$$\begin{aligned} zz_s &= -z + y^{\frac{-2}{3-3n}} \\ &\times \frac{[2(1-3n)^2y - 2(2-3n) - 3(1-n)^2\omega][(1+3n)y+1]}{(1-3n)^2[(1-9n)y-2]} \end{aligned} \qquad (24)$$

where $z_s \equiv dz/ds$. Moreover, with v, given in terms of $v = v(y)$

$$v = \int y^{\frac{-7+3n}{3-3n}} \frac{[2(1-3n)^2y - 2(2-3n) - 3(1-n)^2\omega][(1+3n)y+1]}{3(1-n)(1-3n)^2} dy$$

or, as a function of s

$$v = \int y^{\frac{-2}{3-3n}} \frac{[2(1-3n)^2y - 2(2-3n) - 3(1-n)^2\omega][(1+3n)y+1]}{(1-3n)^2[(1-9n)y-2]} ds$$

converts Eq.(46) to

$$\begin{aligned} zz_v &= 1 - z \times \\ & \frac{(1-3n)^2[(1-9n)y-2]y^{\frac{2}{3-3n}}}{[2(1-3n)^2y - 2(2-3n) - 3(1-n)^2\omega][(1+3n)y+1]} \end{aligned} \qquad (25)$$

where $z_v \equiv dz/dv$. A possible chance to solve Eq.(25) or Eq.(46), emerges if, first, one is be able to express y as an explicit function of either v, or otherwise s,respectively. Note that *****

3 Analytical Solutions

If $k = 0$, the general solution for Eq.(11) is a second degree η polynomial, that is

$$q = A_{0n}\eta^2 + B_{0n}\eta + C_{0n} \qquad (26)$$

with A_n, B_n, and C_n constants where

$$A_{0n} = \frac{1}{2}[2(2-3n) + 3(1-n)^2\omega]m_n \quad , \qquad (27)$$

while B_{0n}, and C_{0n} at this point remain undetermined. Regarding $k=0$ these, the general solutions for Eq.(11), substituted into Eq.(12) turn out not only the cosmic solutions for the flat, ($k=0$), and the curved ($k=\pm1$), spaces but also possibly three different type of solutions for each of the three FRW curvatures exist, each distinguished by the sign of the determinant $\Delta \equiv 4A_{0n}C_{0n} - {B_{0n}}^2$ which, itself, through n depends on the equation of state, and the coupling parameter ω. For each type of determinant, $\Delta > 0$, $\Delta < 0$, and $\Delta = 0$ the scalar field ϕ has two branches: essentially, one with ϕ an increasing function of time and the other with ϕ a decreasing function of time (the solutions were given explicitly and thoroughly discussed by Gurevich et.al. [16], Ruban and Finkelstein [17], see also Morganstern [18]). When $k \neq 0$, their behavior is, in general, considerably more complicated to obtain. Anyhow, it is easy to prove that particular solutions for the curved spaces are special instances of the general solution for the flat space Eq.(26). For $k \pm 0$, the following conditions must be met: On the one hand, once more one gets

$$A_{\pm n} = \frac{1}{2}[2(2-3n) + 3(1-n)^2\omega]m_n$$

however, this time Eq.(11) also imposes the extra conditions

$$(1+3n)A_{\pm n} = -m_n(1-3n)^2$$

in addition to

$$(1+3n)B_{\pm n} = -2(1-3n)\eta_0 \quad ,$$

On the other hand, this solution substituted into Eq.(12) gets

$$B_{\pm n} = \frac{1}{2}[2(2-3n) + 3(1-n)^2\omega]m_n \quad ,$$

and

$$C_{\pm n} = \frac{1}{2}[2(2-3n) + 3(1-n)^2\omega]m_n \quad ,$$

Therefore, if $n = 1/3$, $n = 0$, or when $\eta_0 = 0$, $\Delta_{n\pm} = 0$: It is under this last instance, to which the simple, monotonic in the FRW metric time t, belong. Valid for any curvature, particular power law t time solutions, can be obtained if the scale factor has a manageable in η expression.

Valid for any curvature, these particular power law, t time solutions are almost the only ones that most authors with different purposes usually invoke assuming, in addition, that space is flat. To illustrate these last remarks consider the next concrete example.

For $k=0$, a nowadays relevant solution included within the variety of cosmic possibilities given by the $\Delta_{n\pm} = 0$ case when $\eta_0 = 0$, is shown next (see [12]). From

$$q = A_{n0}\eta^2$$

regarding Eq.(**??**) gets

$$a = a_0 \eta^{\frac{[1+(1-n)\omega]m_{n0}}{A_{n0}}} \quad .$$

Now, $dt = a^{3n} d\eta$ means that

$$t = \frac{2a_0 A_{n0}}{[4+3(1+n)(1-n)\omega]m_{n0}} \eta^{\frac{[4+3(1+n)(1-n)\omega]m_{n0}}{2A_{n0}}}$$

in general. However, another possibility arises if

$$A_{n0} = -3n[1+(1-n)\omega]m_{n0} \tag{28}$$

is met, because then $dt = a^{3n} d\eta$ obtains $\eta \propto e^t$. Besides, also

$$A_{n0} = [4 + 6n + 3(1-n)^2\omega]m_{n0}$$

and both conditions can be satisfied at the same time if

$$\omega = \frac{-4}{3(1-n)(1+n)} = \frac{-4}{3(1-n^2)} \quad .$$

Now, with a a function of t the convention $\dot{a}/a = H = Const.$ here implies $\dot{a}/a = H = -1/3n$ that is

$$a = a_0 e^{Ht} = a_0 e^{-t/3n}$$

so in terms of the original FRW time t, in this case q transforms as

$$q \propto e^{2t},$$

and given that $H = -1/3n$, one rewrites

$$q = q_0 e^{-6nHt}.$$

Furthermore, with $q = \phi a^{3(1-n)}$r r obtains

$$\phi = \phi_0 e^{-3(n+1)Ht} \ , \qquad \phi_0 > 0$$

$$\rho = \rho_0 e^{-nHt} \quad , \qquad p = \left(\frac{n-3}{3}\right)\rho \quad . \qquad \rho_0 > 0$$

n=0 gives us the general relativity case, where ϕ = constant. Here, it turns out that gravity is " asymptotically free " in the sense that the coupling between matter and gravity is initially negligible, one consequence being that, for the scalar metric perturbations produced during this period their amplitud might be suppressed therby eliminating the fine tunning problem of density perturbations [8]. It also has the potential to solve the horizon, flatness and monopole problems. The coupling parameter ω and the integration constants satisfy the following relation in Brans-Dicke theory

$$\omega = \frac{12}{n(n-6)} \quad ,$$

$$\frac{\phi_0 H^2}{8\pi\rho_0} = \frac{(n-6)}{3(2-n)(n-3)}$$

This solution implies that

$$(\ln a)^{\cdot} = H \quad ,$$

and

$$(\ln \phi)^{\cdot} = nH \quad .$$

The deceleration parameter q takes on the permanent value

$$q = -(a)^{\cdot\cdot}\, a/a^{\cdot\, 2} = -1 \quad ,$$

and the age of the universe is infinite. We notice here that a negative value of the deceleration parameter is necessary for inflation, and that in these models it is independent of the equation of state.

The Weak Energy Condition (WEC), Dominant Energy Condition (DEC), and Strong Energy Condition (SEC) for a perfect fluid are:

$$\begin{aligned} WEC: &\quad \rho \geq 0 \text{ and } \rho + p \geq 0, \\ DEC: &\quad \rho \geq 0 \text{ and } -\rho \leq p \leq \rho, \\ SEC: &\quad -\rho \leq p \leq \rho \quad \text{and } \rho + 3p \geq 0 \end{aligned}$$

In this regard the STT inflation solutions, in terms of their predicted energy density-pressure relation, are finally examined. For the solution under consideration the energy conditions impose the following constraints on the parameters:

$$\begin{aligned} &WEC: \rho_0 \geq 0 \text{ and } n \geq 0 \\ &DEC: 0 \leq n \leq 6 \\ &SEC: 2 \leq n \leq 6 \end{aligned}$$

.

The observational results [13] imply that $\omega \geq 3500$, and from eq.(10a) we have, in JBD theory and the scalar tetradic B theory that

If $\omega \geq 0 \Rightarrow n \leq 0$ or $n \geq 6$

If $\omega \leq 0 \Rightarrow 0 \leq n \leq 6$.

For the cosmological model under consideration $\omega \leq 0$ violates only the SEC when $n \leq 2$. For $n \to 0^+$ the equation of state is $p = -\rho$, that is typical of the EGR inflationary era of the universe. In fact, the violation of the strong energy condition is a necessary condition for inflation or generalized inflation to occur, Barrow [14] has even suggested to define inflation as the violation of the strong energy condition. In short we see that a negative ω give us sensible solutions in the JBD with two limiting equations of state: $p = \pm\rho$.

4 FRW Dust Models in BD

Through g instead of q, Eq.(12) for $n = 0$ is

$$\left(\frac{\dot{g}}{g} + \frac{2T}{g}\right)^{\bullet} + \frac{1}{3}\left(\frac{\dot{g}}{g} + \frac{2T}{g}\right)^2 + \omega\left[\frac{m_0}{g} + \left(\frac{T}{g}\right)^2\right] = 0\,. \tag{29}$$

On the other hand, also in terms of g Eq.(11) displays as

$$\ddot{g} - (4 + 3\omega)m_0 = -6\,k\,\exp\left\{\frac{1}{3}\int\left(\frac{\dot{g}}{g} + \frac{2T}{g}\right)dt\right\} \tag{30}$$

The third derivative of g is

$$\begin{aligned}\frac{d^3 g}{dt^3} &= -2k\left(\frac{\dot{g}}{g} + \frac{2T}{g}\right)\exp\left\{\frac{1}{3}\int\left(\frac{\dot{g}}{g} + \frac{2T}{g}\right)dt\right\} \\ &= \frac{1}{3}\left(\frac{\dot{g}}{g} + \frac{2T}{g}\right)\left[\ddot{g} - (4 + 3\omega)m_0\right]\end{aligned} \tag{31}$$

while its fourth time derivative, where the notation g^{IV} is used to stand for it, is

$$\begin{aligned}g^{IV} &\equiv -6k\left(\exp\left\{\frac{1}{3}\int\left(\frac{\dot{g}}{g} + \frac{2T}{g}\right)dt\right\}\right)^{\bullet\bullet} \\ &= \frac{1}{3}\left[\left(\frac{\dot{g}}{g} + \frac{2T}{g}\right)^{\bullet} + \frac{1}{3}\left(\frac{\dot{g}}{g} + \frac{2T}{g}\right)^2\right]\left[\ddot{g} - (4 + 3\omega)m_0\right]\,.\end{aligned} \tag{32}$$

On the other hand, the third and fourth derivatives of g obtained from Eq.(29) are equal to

$$\frac{d^3 g}{dt^3} = \frac{1}{3}\left(\frac{\dot{g}}{g} + \frac{2T}{g}\right)\left[\ddot{g} - (4 + 3\omega)m_0\right] \tag{33}$$

and

$$g^{IV} = -\frac{1}{3}\,\omega\left[\frac{\dot{T}}{g} + \left(\frac{T}{g}\right)^2\right]\left[\ddot{g} - (4 + 3\omega)m_0\right]\,, \tag{34}$$

respectively. By comparing, Eq.(31) and Eq.(33) are identical. However, the last line of Eq.(32) and Eq.(34) are not equal, not even alike. Yet, after they are equated the result is Eq.(29). This scheme brings forward the link that follows between the third, and fourth derivatives of g

$$\left(\frac{\dot{g}}{g} + \frac{2T}{g}\right)g^{IV} = -\omega\left[\frac{\dot{T}}{g} + \left(\frac{T}{g}\right)^2\right]\frac{d^3 g}{dt^3} \tag{35}$$

which turns out to be a constriction for possible solutions to Eq.(30). Considerations that lead to the analytic solutions of Eq.(29) come next, and further on an interpretation regarding its implications in the cosmological context.

5 Linear Fractional Transformations

Assuming $t_0 \neq 0$, and written as

$$3g\ddot{g} - 2\dot{g}^2 - 2T\ddot{g} + (4+3\omega)T^2 + 3(2+\omega)m_0 g = 0 \tag{36}$$

takes the form

$$3{m_0}^2 g g_{_{TT}} - 2{m_0}^2 {g_{_T}}^2 - 2m_0 T g_{_T} + (4+3\omega)T^2 + 3(2+\omega)m_0 g = 0 \tag{37}$$

when the derivatives appearing in Eq.(29) are given in terms of T instead of t, where $g_{_T} \equiv dg/dT$.

A formal solution for Eq.(37) is obtained in terms of the yet undetermined function y in parametric form

$$m_0 g(T) = x \exp 2(\int y(x)dx + Const.)$$
$$T = \exp(\int y(x)dx + Const.) \tag{38}$$

whose corresponding inverse relations are

$$y(x) = \frac{T^2}{m_0(Tg_{_T} - 2g(T))} \tag{39}$$

and

$$x = \frac{m_0 g(T)}{T^2} \tag{40}$$

respectively. Next, through a linear fractional transformation with the two new variables v and u where $v = e^{-2u}g$, and $u = \ln T$, Eq.(37) convertes into

$$3{m_0}^2 v v'' - 2{m_0}^2 {v'}^2 + m_0(m_0 v - 2)v' - (m_0 v + 1)(2m_0 v - 4 - 3\omega) = 0 \tag{41}$$

where $(\ )^{'} \equiv d(\ )/du$. This is a second order equation that does not explicitly contain the independent variable u. Without losing generality, in the above call $m_0 v \equiv x$ to get

$$3xx'' - 2{x'}^2 + (x-2)x' - (x+1)(2x - 4 - 3\omega) = 0 \tag{42}$$

Calling $x' \equiv y$ where $y_x \equiv dy/dx$ converts Eq.(42) into

$$3xyy_x - 2y^2 + (x-2)y - (x+1)(2x - 4 - 3\omega) = 0 \tag{43}$$

which is a first order equation. $y(x)$ is the sought after function appearing in Eq.(38). Eq.(43) can be further simplified, first by letting $y = x^{2/3}z$ which eliminates from it the non linear term, y^2

$$zz_x + \frac{1}{3}(x-2)x^{-5/3}z - \frac{1}{3}(x+1)(2x-4-3\omega)x^{-7/3} \;=\; 0 \tag{44}$$

this is an Abel equation of the Second Kind, and with

$$r = \frac{1}{3}\int (x-2)x^{-5/3}dx$$

Eq.(44) is next turned into

$$zz_r \;=\; -z + \frac{(x+1)(2x-4-3\omega)}{(x-2)x^{2/3}}\,, \tag{45}$$

where $z_r \equiv dz/dr$. Moreover, with

$$s = \frac{1}{3}\int (x+1)(2x-4-3\omega)x^{-7/3}dx$$

Eq.(45) convertes to

$$zz_s \;=\; \left\{\frac{-(x-2)x^{2/3}}{(x+1)(2x-4-3\omega)}\right\} z \,+\, 1 \tag{46}$$

where $z_s \equiv dz/ds$. To have a chance to solve Eq.(45) or Eq.(46), one must first be able to express y as an explicit function of either r, or s, respectively.

Eq.(45) can be written as

$$zz_r \;=\; -z + \frac{(2x-4-3\omega)}{(x-2)}r\,, \tag{47}$$

Therefor $\frac{(2x-4-3\omega)}{(x-2)} = 2$ is a case that obtains, precisely, when $\omega = 0$. Note that in this instance Eq.(44), together with the two aforementioned equations, now turn into

$$zz_x + r_x z - 2r_x r \;=\; 0$$

where the notation

$$r_x = \frac{1}{3}(x-2)x^{-5/3} \tag{48}$$

in it, has been used so that the equations that follow from it are

$$zz_r \;=\; -z + 2r$$

and

$$zz_s \;=\; (-1/2)s^{-1/2}z \,+\, 1\,,$$

respectively. Clearly, a solution for the z_r equation with $\omega = 0$ is

$$z = r \tag{49}$$

which in terms of x, this last as solution for Eq.(44) takes the form

$$z = z(x) = (x+1)x^{-2/3}\,. \tag{50}$$

Again, for $\omega = 0$, and with regard to Eq.(48)

$$z = z(x) = 3xr_x = (x-2)x^{-2/3}\,, \tag{51}$$

is another solution for Eq.(44).

Displayed in terms of g the above results translate into

$$m_0 g = T^2(T-1)\,, \tag{52}$$

and

$$m_0 g = T^2(T+2). \tag{53}$$

Substituted into Eq.(30) both of them land the result

$$m_0 \;=\; -k\,, \tag{54}$$

and

$$a \sim -kt \tag{55}$$

in this case. Further knowledge on the mathematical character of Eq.(42) is extracted in the next section.

6 Differential Equations with Complex Domain

The original investigations undertaken by Painlevé [**?**] near the end of the nineteenth century, and by Gambier [**?**] and others to determine the properties that a non–linear, second order differential equation of polynomial class must enjoy in order for it only to have single-valued solutions without movable singularities other than poles concerns us in this section. The critical points of a solution -branch points and essential singularities- must be fixed instead of beeing movable points. Basically, when its solutions have no essential singularities or branch points depending on the integration constants these equations are said to possess the Painlevé property. They may only have poles as movable singularities. So that these functions are endowed with a special property: Only the members of the class mentioned above possess non movable, that is, fixed critical points in which case its analytical solutions are much easier to obtain. Usually, in addition to poles, the solutions of nonlinear equations have other critcal points such as branch points and essential singularities. Their studies led them to establish a canonical set of fifty equations with the aforementioned property. One would expect on physical grounds Eq.(42) to belong to this canonical set, even if it were still necessary to perform further linear fractional transformations to explicit render it a member of the aforementioned class. Otherwise, to establish integrability conditions one can always perform over a given polynomial class equation a Painlevé test to analyze its general solutions regarding the behavior of its singularities structure in the complex plane [**?**]. The notion "integration of a differential equation" is treacherous. According to Painlevé in the words of Conte: "To integrate an ordinary differential equation is to find a finite expression for the general solution, possibly multivalued, in a finite number of functions, valid in the whole domain of definition" [?]. They were able to reduce every second-order differential equation of the form

$$W'' = F(W', W, X)\,,$$

which is algebraic in the dependent function, rational in its first derivative and analytical in the independent variable into any of fifty canonical equations. Six of these canonical equations define new classes of trascendental meromorphic functions (see E. L. Ince [?], also [?], and [?]) while the rest could be reduced to equations with allready known solutions.

Assuming x is not a constant it was sufficient for our purpose to establish that according to the now standard enumeration of the canonical equations set Eq.(42) is identified to belong to a type number $XXVII$ case only when $\omega = 0$, with its two previously found solutions

$$x = \begin{cases} e^u - 1 \\ e^u + 2 \end{cases} \tag{56}$$

On the other hand, the condition $x = Const.$ sets up two particular solutions for Eq.(37)

$$x = \begin{cases} \frac{1}{2}(4 + 3\omega) \\ \quad -1 \end{cases} \tag{57}$$

Through the relation $g = {m_0}^{-1} x T^2$ the first, corresponds to the leading coeficient of the general, analytic solution for the flat space: a second degree time polynomial obtained by integrating Eq.(30)

$$g = g_0 = A_0 t^2 + B_0 t + C_0 \tag{58}$$

with

$$A_0 = \frac{1}{2}(4 + 3\omega) m_0$$

This solution substituted into Eq.(36) yields the relation between the constants A_0, B_0, C_0, and t_0

$$3(3 + 2\omega) m_0 C_0 = {B_0}^2 + t_0 B - A_0 {t_0}^2$$

however B_0 and t_0 remain undetermined, and are therefore free parameters so that the flat space is able to expand in distinct ways determined through the sign of the discriminant $4A_0C_0 - {B_0}^2 \equiv \Delta$

$$\Delta \;=\; \frac{[(4 + 3\omega)t_0 - B_0]^2}{3(3 + 2\omega)} \tag{59}$$

Beside, the scalar field ϕ can evolve in any one of two opposite ways: in a growing or in a decresing mode (through different procedures these solutions were first obtained, and discussed by Weinberg [?], Gurevich et.al. [?], Ruban and Finkelstein [?], see also Morganstern [?], and Nariai [?]).

Meanwhile, the second relation for x substituted into $g = {m_0}^{-1}xT^2$ gives way to two particular solutions, one for the closed, and the other for the open space. Indeed, for Eq.(30) $m_0gT^{-2} = -1$ is a possibility only if $k \neq 0$, and similar to the flat space case it also turns out a second degree time polinomial. However, this time polinomial substituted into Eq.(29) obtains as its more general form the expresion

$$g = -m_0t^2 - 2t_0t - 2\frac{{t_0}^2}{m_0} = -{m_0}^{-1}(m_0t + t_0)^2 \tag{60}$$

where here t_0 is the only free available parameter. Therefore this solution is less general than the corresponding flat space expresion. Now, from Eq.(**??**) putting $t_0 = 0$ the scale factor obtains $a = a_0t$ without loosing generality, and specifically from Eq.(30) one gets

$$a = \left(\frac{-2k}{2+\omega}\right)^{\frac{1}{2}} t\,. \tag{61}$$

For it to be real

$$k = -1 \Rightarrow \ \omega > -2 \tag{62}$$

or

$$k = 1 \Rightarrow \ \omega < -2 \quad . \tag{63}$$

Both curved spaces expand linearly. This means that the negative curvature space is actually a Minkowski space expressed in expanding coordinates. On the other hand, when $k = 1$ a linear expansion does not lead to a Milne type solution. It has been argued that negative ω values leads to unacceptable physical results [**?**]. Bautista et al [**?**] consider possible negative values as long as $\omega > -2$, and under certain conditions String theory [**?**] predicts an $\omega = -1$ low energy effective value.

7 Radiation

The equation for radiation or an ultrarelativistic fluid $p = \frac{1}{3}\rho$ implies

$$\rho \ = \ M_r a^{-4}, \qquad M_r = const.$$

such a fluid cannot generate a scalar field that if present must then be sourceless

$$\dot{\phi} a^3 \ = \ \gamma = const. \neq 0 \quad , \tag{64}$$

here $r \equiv \phi a^2$ depends on variable "η" defined by $dt \equiv a d\eta$.

With these variables one has the following set of equations: With

$$\frac{\phi'}{\phi} \ = \ \frac{\gamma}{r} \quad , \tag{65}$$

one has

$$2\frac{a'}{a} \ = \ \frac{r'}{r} - \frac{\gamma}{r} \tag{66}$$

so that

$$a^2 = r \exp\left(-\int \frac{\gamma}{r} dt\right) \ . \tag{67}$$

The dynamic equation is

$$\frac{r''}{r} = \frac{2}{3}\frac{m_r}{r} \ - \ 4k \quad , \qquad k = 0, \ \ \pm 1 \tag{68}$$

with its constriction give by

$$\left(\frac{r'}{r}\right)^2 - \frac{3+2\omega}{3}\left(\frac{\gamma}{r}\right)^2 = \frac{4m_r}{3r} - 4k \quad , \qquad m_r \equiv 8\pi M_r \tag{69}$$

These two equations don't contain terms with the scale factor variable a even if $k \neq 0$ that in the equations for other fluids is present. This fact makes it unnecessary to recourse to a third order differential equation in order to eliminate the aformentioned term so that here only the dependent variable r appears.

When $k = 0$ Eq.(68), and Eq.(69) obtain

$$r = R\eta^2 + Q\eta + P \ , \qquad R\, , Q \, and \, P \ = const. \tag{70}$$

with $R = \frac{1}{3}m_r$, while $3Q^2 \ = \ 4m_r P + (3+2\omega)\gamma^2$ P remains free. The determinant is

$$\Delta = -\frac{(3+2\omega)}{3}\gamma$$

when $\Delta = 0$ one obtains the corresponding Einstein's solution. In all three different cosmic solutions for this FRW flat space ($k = 0$) model exist distinguished by a determinant whose sign depends on ω. The solutions have been thoroughly discussed by Gurevich et.al. [16], Ruban and Finkelstein [17], (see also Morganstern [18]. For $k \neq 0$ Eq.(68) and Eq.(69) have the following solutions

$$
\begin{aligned}
\eta + \eta_0 \quad &= \\
&\tfrac{1}{2} \quad \ln 2(16k^2r^2 + \frac{16}{3}m_r r + \frac{(3+2\omega)}{3}\gamma^2)^{\frac{1}{2}} - 8kr + \frac{4}{3}m_r \\
&\tfrac{1}{2} \quad arsh\frac{-8kr + \frac{4}{3}m_r}{\Delta^{\frac{1}{2}}} \\
&\tfrac{-1}{2} \quad \arcsin\frac{-8kr + \frac{4}{3}m_r}{-\Delta^{\frac{1}{2}}} \\
&\tfrac{1}{2} \quad \ln(-8kr + \frac{4}{3}m_r)
\end{aligned}
\tag{71}
$$

the first for $k = -1$, $k = -1$ and $\Delta > 0$ for the second, $k = 1$ and $\Delta < 0$ for the third, and $k = -1$ and $\Delta = 0$ for the last with $\Delta = -\frac{16}{9}(m_r + 3(3+2\omega)\gamma^2$.

Moreover when $\Delta \neq 0$, for each type of determinant one one can get two solutions depending on the sign of ω. Other particular solutions for $k \neq 0$ are

$$r = r_0 = const. \tag{72}$$

with $r_0 = \frac{1}{6k}m_r$, so that

$$a = \left(\frac{1}{6k}m_r\right)^{\frac{1}{2}} \exp\left(-\frac{k\gamma}{3m_r}\eta\right) . \tag{73}$$

This last expression can be given in terms of the "*cosmictimet*. The result is a linear expansion

$$a = \left(\frac{6k}{m_r}\right)^{\frac{1}{2}} t = \left(-\frac{3k}{3+2\omega}\right)^{\frac{1}{2}} t \ , \tag{74}$$

because in this case

$$[-3(3+2\omega)\gamma^2]^{\frac{1}{2}} t \ , \tag{75}$$

Other particular solutions can be obtained when $\gamma = 0$ which are also solutions for the Einstins corresponding models.

4. VACUUM

The vacuum field equation are similar to the ones corresponding to the radiation model because the scalar field for this case must also be sourceless. With the notation $v \equiv \phi a^3$, these are

$$\frac{\dot{\phi}}{\phi} = \frac{\nu}{v} \ , \qquad \nu = const. \tag{76}$$

from which the relation between a and v

$$3\frac{\dot{a}}{a} = \frac{\dot{v}}{v} - \frac{\nu}{v} \ , \tag{77}$$

is obtained so that

$$a^3 = v \exp\left(\frac{-\nu}{3}\int\frac{dt}{v}\right) . \tag{78}$$

For $\rho = p = 0$, the dynamic equation is

$$\frac{\ddot{v}}{v} \;=\; -6\,\frac{k}{a^2} \;\;, \qquad\qquad k = 0,\;\; \pm 1 \tag{79}$$

and its constriction, after the curvature term is replaced by Eq.(79)

$$3\frac{\ddot{v}}{v} - 2\left(\frac{\dot{v}}{v}\right)^2 - 2\left(\frac{\nu}{v}\right)\frac{\dot{v}}{v} + (4+3\omega)\left(\frac{\nu}{v}\right)^2 \;=\; 0 \;. \tag{80}$$

A simple inspection makes it clear that the above equations are directly integrable: For $k = 0$

$$v \;=\; ct \;+\; t_0 \;\;, \qquad\qquad c\,, t_0 = const. \tag{81}$$

where

$$2c \;=\; \left[-1 \pm (3(3+2\omega))^{\frac{1}{2}}\right]\nu \;.$$

For $k \neq 0$, the use of $dt = v d\tau$ hands out for $k = 1$

$$v \;=\; \left[\cosh\left(\frac{(-\Delta)^{\frac{1}{2}}}{6}\tau\right)\right] e^{-\frac{\nu}{2}(\tau+\tau_0)} \quad ,\omega > -3/2 \tag{82}$$

and for $k = -1$

$$v \;=\; \left[\cos\left(\frac{(\Delta)^{\frac{1}{2}}}{6}\tau\right)\right] e^{-\frac{\nu}{2}(\tau+\tau_0)} \quad ,\omega < -3/2 \tag{83}$$

where $\Delta \equiv -12(3+2\omega)\nu^2$. The corresponding scale factors for the above two are

$$a \;=\; \left[e^{-\frac{\nu}{2}(\tau+\tau_0)}\cosh\left(\frac{(-\Delta)^{\frac{1}{2}}}{-6}(\tau+\tau_0)\right)\right]^{\frac{1}{2}} \quad , \tag{84}$$

when $k = 1$, $\omega > -3/2$ and for $k = -1$, $\omega < -3/2$

$$a \;=\; \left[e^{\frac{\nu}{2}(\tau+\tau_0)}\cos\left(\frac{(\Delta)^{\frac{1}{2}}}{6}(\tau+\tau_0)\right)\right]^{\frac{1}{2}} \quad . \tag{85}$$

8 CONCLUSIONS

The general evolution of BD dust in curved space models was obtained from the analysis concerning its “Painlevé properties”. Specifically, by regarding the properties of a second order nonlinear differential equation of polynomial class. The unfolding of Friedman–Lemaître Einstein’s dust models are described through the scale factor function only. Two time dependent variables, the scale factor and the scalar field, together with a free parameter mark the evolution of the corresponding BD cosmology. However, it is shown that these two variables subject to a constriction as a single function entirely capable to characterize by itself through a set of non–linear equations the unfolding of the latter models. The new form in which the equation is written representes a clear, more direct, and powerful way to obtain its analytical solutions (see [**?**]).

Distinct corners of astronomy now suggest that amongst other properties, our universe is flat. Within the context of Einsteins General Relativity theory the curvature of spacetime is determined by the ammount of its energy content, and in its Friedman–Lemaître models cosmologists with the aid of particle physics try to explain or justify not only how different epochs come about to conform the evolution of the universe but why its mass–energy density content is almost precisely equal to its critical value to render it flat. On the other hand, the dust FRW flat space model in BD singles itself out from the other spaces. Another significant fact which, it appears, Gurevich et.al. [**?**] were the first to make evident is that the scalar field ϕ in its ”lowest mode” has a sourcless charecter meaning that at this level it has no connection with ordinary matter so it does not satisfy Mach’s Principle, and therefore the possibility of having in this, the Jordan frame, a nonminimal coupled scalar field not related to ordinary matter is noteworthy in view of the now present “dark matter” and “dark energy” conjecture. If one finds that the most simple, limited and particular BD case, does not give adecuate, alternative and acceptable explanations to the cosmological outcomes predicted by GR still at present held by many to be undisputed. Cosmic observations suggests one to reconsider under a different light the role that a non minimally scalar field might play in some other scalar–tensor theories. The BD results shows that the above possibility is worthwhile to explore.

Acknowledgements

This work was supported by CONACyT Grant No. 5-3672-E9312

References

[1] C. Brans and R. H. Dicke, *Phys. Rev.* **124** 925 (1961); P. G. Bergmann, *Int. J. Theor. Phys.* **1** 25 (1968); R. V. Wagoner, *Phys. Rev. D* **1** 3209 (1970); K. Nordvedt, *Astrophys. J.* **161** 1059 (1970).

[2] P. Chauvet, *Astrophys. Space Sci.* **90** 51 (1983).

[3] J. Benítez, A. Macías, and E. W. Mielke,

[4] P. Chauvet, J. L. Cervantes-Cota, and H. N. Núñez-Yépez, in *Class and Quantum Grav.* **9**, 1923 (1992).

[5] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum *Exact Solutions of Einstein's Field Equations*, ed E. Schmutzer (Cambridge: Cambridge University Press, 1980).

[6] M. A. H. MacCallum, in *General Relativity. An Einstein Centenary Survey*, ed S W. Hawking and W. Israel (Cambridge: Cambridge University Press, 1980).

[7] V. A. Belinskii, E. M. Lifschitz, and I. M. Khalatnikov, *Sov. Phys.: Usp* **13**, 475; and *Adv. Phys,* **19**, 525 (1970).

[8] C. W. Misner, *Phys. Rev. Lett.* **22**, 1071 (1969).

[9] P. Chauvet and O. Pimentel, *Gen. Rel. Grav.* **24**, 243 (1992).

[10] P. Chauvet and J. L. Cervantes-Cota, and H. N. Núñez-Yépez, in *Proceedings of the 7 th Latin American Symposium on General Relativity and Gravitation, SILARG VII* World Scientific, 1991, p. 487.

[11] Dehnen and O. Obregón, *Astrophys. Space Sci.* **14**, 454 (1971); P. Chauvet and O. Obregón, *Astrophys. Space Sci.* **66**, 515 (1979).