Remodeling the B-model: Implications for large N duality

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December 15th, 2007
Miami

Never again will a single story be told
as if it is the only one.

- *John Berger*

Punch line

- We propose a new, complete, B-model formalism to compute open and closed topological string amplitudes on mirrors of toric Calabi-Yau threefolds
- The formalism is nonperturbative in the moduli, hence can be used to do precise comparisons with the perturbative gauge theory obtained through large N duality

References:

- V.B., A. Klemm, M. Mariño and S. Pasquetti, 0709.1453
- V.B., A. Klemm, M. Mariño, S. Pasquetti and M. Weiss, work in progress
- M. Mariño, hep-th/0612127
- B. Eynard and N. Orantin, math-ph/0702045

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Why is that interesting?

- Various mathematical applications:
  - (Open) orbifold Gromov-Witten theory [BKMP]
  - Relation with matrix models and nonperturbative approach to topological strings [Mariño,EMO,MSW]
  - Hodge integrals, Hurwitz numbers, . . . [BM]

- Topological strings/Chern-Simons theory provides a toy model of AdS/CFT
  - Using our formalism, we can test this duality in very much detail in the perturbative regime of Chern-Simons theory [BKMP]

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Outline

1. Large N duality in topological strings
2. New approach to B-model topological strings
3. Implications for large N duality

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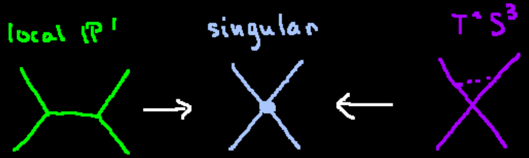
Gopakumar-Vafa correspondence

Start with the geometric transition:

$$\text{local } \mathbb{P}^1 \longrightarrow \text{conifold} \longleftarrow T^*S^3$$



Consider A-model topological string theory on both sides:

LHS: Closed topological strings on local $\mathbb{P}^1$
$\rightarrow$ defined as a perturbative expansion in e^{-t} at large radius $t \rightarrow \infty$ (t is the Kähler parameter of $\mathbb{P}^1$)

RHS: Open topological strings on T^*S^3 with N D-branes wrapped around S^3
$\rightarrow$ Worldvolume gauge theory is $U(N)$ Chern-Simons theory on S^3 [Witten]

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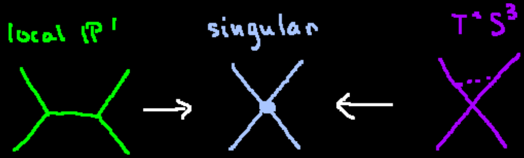

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Gopakumar-Vafa correspondence

The main statement

Strings on local $\mathbb{P}^1 \leftrightarrow U(N)$ Chern-Simons on S^3

- Kähler parameter $t \leftrightarrow$ 't Hooft parameter of CS
- Perturbative CS in $t \leftrightarrow$ expansion of topological strings at small radius $t \to 0$ ...we don't know how to do that in general!
- Topological statement of weak/strong duality

- We can study CS non-perturbatively [Witten], and compare with topological strings at large radius [GV]
- We can try to define A-model at small radius and compare with perturbative CS (goal of this talk)

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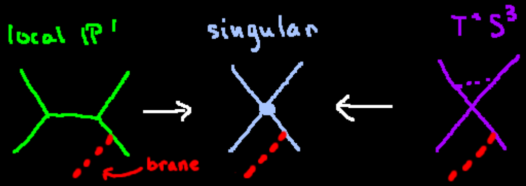
Consider probe D-branes in local $\mathbb{P}^1$, which are pushed through the geometric transition:



What we get is:

Open version

Open strings on local $\mathbb{P}^1$ $\leftrightarrow$ Wilson loops in CS on S^3 [OV]

Generalization: quotienting Gopakumar-Vafa

Quotient both sides of the transition by $\mathbb{Z}_2$, and resolve the fixed point:

$$\text{local } \mathbb{P}^1 \times \mathbb{P}^1 \to \text{singular} \leftarrow T^*(S^3/\mathbb{Z}_2)$$

We get:

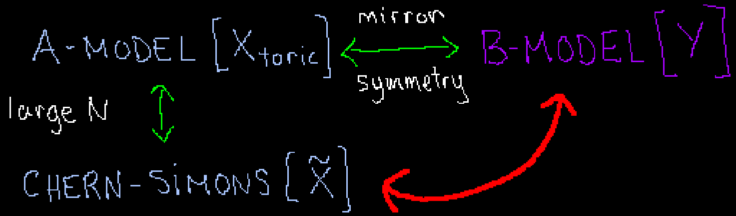
Strings on local $\mathbb{P}^1 \times \mathbb{P}^1 \leftrightarrow U(N)$ CS on $S^3/\mathbb{Z}_2$ [Mariño,AKMV]

- Can also consider $\mathbb{Z}_p$ quotients
  $\Rightarrow A_{p-1}$ fibration over $\mathbb{P}^1 \leftrightarrow$ lens space $L(p,1)$
- Open strings $\leftrightarrow$ Wilson loops in CS theory
- Interesting to study more general threefolds than just local $\mathbb{P}^1$
  $\rightarrow$ toric Calabi-Yau threefolds

Topological strings at small radius

- Difficult to study A-model at small radius directly
- Use mirror symmetry to map the problem to the B-model, which is nonperturbative in the moduli, hence can be expanded at small radius

We want a complete formalism to compute B-model amplitudes nonpertubatively in the moduli, for the mirrors threefolds to toric Calabi-Yau threefolds



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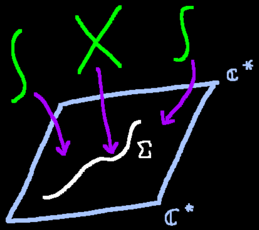
Mirror pairs (X, Y) of Calabi-Yau threefolds

1. X: toric Calabi-Yau threefold (noncompact)
   $\longrightarrow$ Examples: local $\mathbb{P}^1$, local $\mathbb{P}^1 \times \mathbb{P}^1$, local $\mathbb{P}^2$, ...
2. $Y : \{ww' = H(x, y)\} \subset \mathbb{C}^2 \times (\mathbb{C}^*)^2$
   $\longrightarrow$ conic fibration over $\mathbb{C}^* \times \mathbb{C}^*$, such that the conic fiber degenerates to 2 lines over a Riemann surface (the mirror curve)

$$\Sigma : \{H(x, y) = 0\} \subset (\mathbb{C}^*)^2.$$

Σ encodes the mirror geometry Y.



Open mirror symmetry

1. noncompact probe brane in X
   $\longrightarrow$ dual to Wilson loops in corresponding CS theory
2. brane wrapping one of the two lines over Σ
   $\longrightarrow$ moduli space of the brane is the mirror curve Σ

So we want to study:

Open B-model topological strings on Y of the form above with branes wrapping one of the two lines over Σ

- The open B-model amplitudes should live on the mirror curve Σ, since this is the open/closed moduli space (the moduli space of vacua of the theory)
- We want to build a set of open and closed amplitudes living on a Riemann surface Σ

Matrix models!

Suppose that Σ is the spectral curve of a matrix model.

- Loop equations of the matrix model $\Rightarrow$ recursive solution for the free energies F_g and the correlation functions $W_k^{(g)}$ of the matrix model [EO]
- F_g: functions on Σ
  $W_k^{(g)}$: meromorphic differentials on Σ

The recursion solution is entirely geometric on Σ
$\Rightarrow$ can be defined for any Σ, whether it is the spectral curve of a matrix model or not
$\Rightarrow$ what do the F_g and $W_k^{(g)}$ mean?

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Our claim

When Σ is a mirror curve, the F_g are the genus g closed B-model amplitudes, while the $W_k^{(g)}$ are the genus g, k holes open B-model amplitudes.

Why?

- In [Mariño,BKMP], we checked many many examples, and it works :-)
- In [DV], a sketch of a proof of the recursion from Kodaira-Spencer theory is proposed
- From large N duality, for some toric geometries there are CS duals, which can be expressed as matrix models [Mariño], with spectral curves given by the mirror curve

A proof of our claim, either directly in the B-model or through a matrix model dual, is still missing.

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The recursion: Ingredients

Start with a $\Sigma : \{H(x,y)=0\} \in \mathbb{C}^* \times \mathbb{C}^*$.

1. the ramification points $q_i \in \Sigma$ of the x-projection $\Sigma \to \mathbb{P}^1$:

$$\frac{\partial H}{\partial y}(q_i) = 0$$

   - near q_i, there are 2 points $q, \bar{q}$ with $x(q) = x(\bar{q})$

2. the disk amplitude (different from EO because of $\mathbb{C}^* \times \mathbb{C}^*$)

$$\tilde{W}_1^{(0)}(x) = \omega(x) = \log y(x) \frac{\mathrm{d}x}{x}$$

3. the annulus amplitude

$$\tilde{W}_2^{(0)}(x_1,x_2) = B(x_1,x_2) - \frac{\mathrm{d}x_1 \mathrm{d}x_2}{(x_1-x_2)^2},$$

where $B(x_1,x_2)$ is the Bergman kernel of Σ.

The recursion: Bergman kernel

The Bergman kernel of Σ is the unique meromorphic differential $B(x_1, x_2)$ with a double pole at $x_1 = x_2$ with no residue and no other pole, and normalized such that

$$\oint_{A_I} B(x_1, x_2) = 0,$$

where (A_I, B^I) is a canonical basis of cycles for Σ.

- example: if Σ has genus 0, in local coordinate z:

$$B(p, q) = \frac{\mathrm{d}z(p)\mathrm{d}z(q)}{(z(p) - z(q))^2}$$

- define also locally near a q_i:

$$dE_q(p) = \frac{1}{2}\int_q^{\bar{q}} B(\xi, p)$$

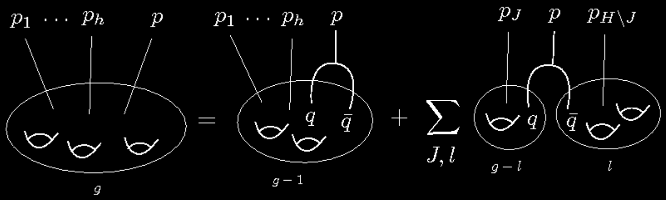
The recursion: First step

Fix:

$$W_1^{(0)}(p_1) = 0, \qquad W_2^{(0)}(p_1, p_2) = B(p_1, p_2).$$

The other differentials are generated recursively by

$$W_g(p, p_1 \cdots, p_h) = \sum_{q_i} \operatorname*{Res}_{q=q_i} \frac{\mathrm{d}E_q(p)}{\omega(q)} \Big[W_{g-1}(q, \bar{q}, p_1, \cdots, p_h) + \sum_{l=0}^{g} \sum_{J \subset H} W_{g-l}(q, p_J) W_l(\bar{q}, p_{H \setminus J}) \Big]$$



The recursion: second step

Let $\phi(p)$ be an arbitrary anti-derivative of $\omega(p)$ (i.e. $\mathrm{d}\phi = \Phi$). Then

$$F_g = \frac{1}{2-2g} \sum_{q_i} \operatorname*{Res}_{q=q_i} \phi(q) W_1^{(g)}(q)$$

Our claim

- F_g: genus g closed B-model amplitudes
- $A_h^{(g)} = \int W_h^{(g)}$: genus g, h hole open B-model amplitudes

Summary so far

- Complete formalism to generate open/closed B-model amplitudes (reproduces topological vertex — A-model — results at large radius)
- Only need the disk and the annulus amplitudes, generate everything else recursively
- Gluing formalism for topological string amplitudes

Main point for this talk

Formalism is valid at any point in moduli space, hence can be used to compute amplitudes at the point mirror to the small radius point.

Outline

Comparing with perturbative Chern-Simons theory

Highly non-trivial example:

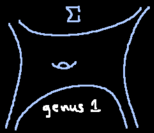
Open strings on local $\mathbb{P}^1 \times \mathbb{P}^1 \leftrightarrow$ Wilson loops in CS on $S^3/\mathbb{Z}_2$

1. Open strings on local $\mathbb{P}^1 \times \mathbb{P}^1$ at small radius
   $\rightarrow$ go to the mirror B-model and use our formalism
2. Perturbative expansion of expectation values of Wilson loops in CS on $S^3/\mathbb{Z}_2$
   $\rightarrow$ use the matrix model representation of CS theory

Topological string side

Mirror curve to local $\mathbb{P}^1 \times \mathbb{P}^1$ (2D moduli space):

$$\Sigma : \{H(x,y) = x^2 y + xy^2 + xy + q_1 y + q_2 x = 0\}$$



Define coordinates [AKMV]

$$x_1 = 1 - \frac{q_1}{q_2}, \qquad x_2 = \frac{1}{\sqrt{q_2}(1 - \frac{q_1}{q_2})}$$

Small radius point: $x_1, x_2 \to 0$.

Generating the amplitudes

- Disk amplitude: just a function, so simply expand around $x_1, x_2 = 0$
- Annulus amplitude: Bergman kernel is quasi-modular, hence transforms with a shift under modular transformations
  - Large radius $\longrightarrow$ small radius is an S-transformation (exchanging the A and B periods)
  - From quasi-modular properties we extract $B^{orb}(x_1, x_2)$ from large radius Bergman kernel
- Once we have disk and annulus, we generate everything else at the small radius point!

Chern-Simons theory on $S^3/\mathbb{Z}_2$

Given by a two-matrix model [Mariño,AKMV]:

$$Z(N_1, N_2, g_s) = \int \mathrm{d}M_1 \mathrm{d}M_2 \exp\Big(-\frac{1}{2g_s}(\mathrm{Tr} M_1^2 + \mathrm{Tr} M_2^2) + V(M_1) + V(M_2) + W(M_1, M_2)\Big)$$

with

$$V(M) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{B_{2k}}{k(2k)!} \sum_{s=0}^{2k} (-1)^s \binom{2k}{s} \mathrm{Tr} M^s \mathrm{Tr} M^{2k-s}$$

$$W(M_1, M_2) = \sum_{k=1}^{\infty} \frac{B_{2k}(2^{2k}-1)}{k(2k)!} \sum_{s=0}^{2k} (-1)^s \binom{2k}{s} \mathrm{Tr} M_1^s \mathrm{Tr} M_2^{2k-s}$$

(the matrix model can be written in a simpler form using Log potentials)

Expectation value of the unknot

To compare with open topological string amplitude we need the expectation value of the unknot in Chern-Simons theory on $S^3/\mathbb{Z}_2$:

$$W_R(N_1, N_2, g_s) = \frac{1}{Z(N_1, N_2, g_s)} \langle \mathrm{Tr}_R \mathrm{e}^M \rangle,$$

where, in terms of eigenvalues m_i^1, m_j^2 of M_1, M_2

$$\mathrm{e}^M = \mathrm{diag}(\mathrm{e}^{m_1^1}, \ldots, \mathrm{e}^{m_{N_1}^1}, -\mathrm{e}^{m_1^2}, \ldots, -\mathrm{e}^{m_{N_2}^2})$$

In this two-matrix model this can be computed perturbatively in the 't Hooft variables $S_i = g_s N_i$, $i = 1, 2$

It works! :-)

Then, we can check that [BKMP]:

Perturbative expansion of the unknot in CS on $S^3/\mathbb{Z}_2$

$$=$$

Open A-model amplitudes of local $\mathbb{P}^1 \times \mathbb{P}^1$ at small radius

On the other hand:

Resummation at strong 't Hooft coupling of CS on $S^3/\mathbb{Z}_2$

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- The resummation involves a complicated moduli space (the complex structure moduli space of the mirror of local $\mathbb{P}^1 \times \mathbb{P}^1$, or the stringy Kähler moduli space of local $\mathbb{P}^1 \times \mathbb{P}^1$), and a clever parameterization is need (choice of flat coordinates)

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Open questions

Can we learn anything about AdS/CFT from this toy model?

highly curved string target $\leftrightarrow$ perturbative gauge theory
large radius string theory $\leftrightarrow$ strong coupling resummation

So we have a complete B-model formalism for mirrors to toric geometries, which is nonperturbative in the moduli.
Many open questions:

- Proof of the recursion in the B-model? (beyond [DV])
- Matrix model duals for topological strings on all toric geometries?
- Compact threefolds?

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