Thomas precession and all that The rest frame result

$$\frac{d}{dt_{\text{rest}}}\boldsymbol{s} = \frac{gq}{2m}\boldsymbol{s}\times\boldsymbol{B}_{\text{rest}}$$

becomes

$$\frac{d}{d\tau}S^\mu = \frac{gq}{2m}\left[F^{\mu\nu}S_\nu - \frac{1}{c^2}U^\mu\left(U_\nu F^{\nu\lambda}S_\lambda\right)\right] - \frac{1}{c^2}U^\mu\left(S_\nu\frac{d}{d\tau}U^\nu\right) \tag{1}$$

when expressed in terms of the axial vector S^μ obtained by transforming from the rest frame

$$\begin{aligned} S^0 &= \gamma\boldsymbol{\beta}\cdot\boldsymbol{s} \\ \boldsymbol{S} &= \boldsymbol{s} + \frac{\gamma^2}{1+\gamma}\left(\boldsymbol{\beta}\cdot\boldsymbol{s}\right)\boldsymbol{\beta} \end{aligned}$$

By definition, the axial vector therefore obeys the constraint

$$U_\mu S^\mu = 0$$

where U^μ is the particle's four-velocity

$$U^\mu = \gamma\left(c, \boldsymbol{v}\right) \ , \quad U_\nu U^\nu = c^2$$

We immediately check the constraint is an invariant using (1). That being so, we may rewrite (1) as

$$\frac{d}{d\tau}S^\mu - \frac{1}{c^2}U^\mu\left(U_\nu\frac{d}{d\tau}S^\nu\right) = \frac{gq}{2m}\left[F^{\mu\nu}S_\nu - \frac{1}{c^2}U^\mu\left(U_\nu F^{\nu\lambda}S_\lambda\right)\right] \tag{2}$$

That is to say, the evolution of the spin axial vector in the direction Lorentz-orthogonal to the four-velocity is just the corresponding orthogonal projection of $\frac{gq}{2m}F^{\mu\nu}S_\nu$.

When the evolution of the four-velocity is purely electromagnetic

$$\frac{d}{d\tau}U^\nu = \frac{q}{m}F^{\mu\nu}U_\nu$$

(1) becomes

$$\frac{d}{d\tau}S^\mu = \frac{gq}{2m}\left[F^{\mu\nu}S_\nu - \frac{1}{c^2}\left(1-\frac{2}{g}\right)U^\mu\left(U_\nu F^{\nu\lambda}S_\lambda\right)\right] \tag{3}$$

In terms of rest frame spin, but laboratory time, this all boils down to Thomas' original form (after massaging) of the equations

$$\gamma\frac{d}{dt}\boldsymbol{s} = \frac{d}{d\tau}\boldsymbol{s} = \boldsymbol{T}_{\text{rest}} + \boldsymbol{\omega}_{\text{Thomas}}\times\boldsymbol{s}$$

where the rest frame torque and Thomas precession rate are

$$\begin{aligned} \boldsymbol{T}_{\text{rest}} &= \boldsymbol{F} - \frac{\gamma}{1+\gamma}F^0\boldsymbol{\beta} \\ \boldsymbol{\omega}_{\text{Thomas}} &= \frac{\gamma^2}{1+\gamma}\left(\frac{d}{d\tau}\boldsymbol{\beta}\right)\times\boldsymbol{\beta} \end{aligned}$$

Note that

$$\frac{d}{d\tau}\boldsymbol{\beta} = \gamma \frac{d}{dt}\boldsymbol{\beta} = \frac{q}{m}\left[\boldsymbol{E} - (\boldsymbol{\beta}\cdot\boldsymbol{E})\,\boldsymbol{\beta} + \boldsymbol{\beta}\times\boldsymbol{B}\right]$$

in terms of the lab frame electromagnetic fields.

Another way to view the whole business is in terms of the spin tensor $S^{\mu\nu} = -S^{\nu\mu}$ defined by

$$\begin{aligned}
S^{\mu\nu} &= \left(\frac{1}{c}\right)\varepsilon^{\mu\nu\alpha\beta}U_\alpha S_\beta\ , \quad U_\mu S^{\mu\nu} = 0 \\
S^{\mu} &= \left(\frac{1}{2c}\right)\varepsilon^{\mu\nu\lambda\sigma}U_\nu S_{\lambda\sigma} \\
S^{\mu}\varepsilon_{\mu\alpha\beta\gamma} &= \left(\frac{-1}{2c}\right)\delta^{\nu\lambda\sigma}_{\alpha\beta\gamma}U_\nu S_{\lambda\sigma} = \left(\frac{-1}{c}\right)\left(U_\alpha S_{\beta\gamma} + U_\beta S_{\gamma\alpha} + U_\gamma S_{\alpha\beta}\right)
\end{aligned}$$

In the particle's rest frame (with $S \to S_{\text{rest}} \equiv s$)

$$s^i = \left(\frac{-1}{2}\right)\varepsilon^{ijk}s_{jk}\ , \quad s_{jk} = -\varepsilon^{ijk}s^i$$

From $\frac{d}{d\tau}U^2 = 0$, and

$$\frac{d}{d\tau}S^\mu = \frac{gq}{2m}\left[F^{\mu\nu}S_\nu - \frac{1}{c^2}U^\mu\left(U_\nu F^{\nu\lambda}S_\lambda\right)\right] - \frac{1}{c^2}U^\mu\left(S_\nu\frac{d}{d\tau}U^\nu\right)$$

this spin tensor evolves as

$$\begin{aligned}
\frac{d}{d\tau}S^{\mu\nu} &= \left(\frac{1}{c}\right)\varepsilon^{\mu\nu\alpha\beta}\left(U_\alpha\frac{d}{d\tau}S_\beta + S_\beta\frac{d}{d\tau}U_\alpha\right) \\
&= \left(\frac{gq}{2mc}\right)\varepsilon^{\mu\nu\alpha\beta}U_\alpha F_{\beta\gamma}S^\gamma + \left(\frac{1}{c}\right)\varepsilon^{\mu\nu\alpha\beta}S_\beta\frac{d}{d\tau}U_\alpha \\
&= \left(\frac{gq}{2mc}\right)\varepsilon^{\mu\nu\alpha\beta}U_\alpha F_\beta{}^\gamma\left(\frac{1}{2c}\right)\varepsilon_{\gamma\rho\sigma\tau}U^\rho S^{\sigma\tau} + \left(\frac{1}{c^2}\right)\left(U^\mu S^{\nu\alpha} + U^\nu S^{\alpha\mu} + U^\alpha S^{\mu\nu}\right)\frac{d}{d\tau}U_\alpha \\
&= \left(\frac{gq}{4mc^2}\right)\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\gamma\rho\sigma\tau}U_\alpha F_\beta{}^\gamma U^\rho S^{\sigma\tau} + \left(\frac{1}{c^2}\right)\left(U^\mu S^{\nu\alpha} + U^\nu S^{\alpha\mu}\right)\frac{d}{d\tau}U_\alpha
\end{aligned}$$

That is to say

$$\frac{d}{d\tau}S^{\mu\nu} = \left(\frac{gq}{4mc^2}\right)\left(-\delta^{\mu\nu\alpha\beta}_{\gamma\rho\sigma\tau}\right)U_\alpha F_\beta{}^\gamma U^\rho S^{\sigma\tau} + \left(\frac{1}{c^2}\right)\left(U^\mu U_\alpha\frac{d}{d\tau}S^{\alpha\nu} + U^\nu U_\alpha\frac{d}{d\tau}S^{\mu\alpha}\right)$$

or

$$\frac{d}{d\tau}S^{\mu\nu} - \left(\frac{1}{c^2}\right)\left(U^\mu U_\alpha\frac{d}{d\tau}S^{\alpha\nu} + U^\nu U_\alpha\frac{d}{d\tau}S^{\mu\alpha}\right) = \mathbb{F}^{\mu\nu}_{\alpha\beta}S^{\alpha\beta}$$

where

$$\mathbb{F}^{\mu\nu}_{\sigma\tau} = \left(\frac{gq}{4mc^2}\right)\left(-\delta^{\mu\nu\alpha\beta}_{\gamma\rho\sigma\tau}\right)U_\alpha F_\beta{}^\gamma U^\rho$$

Ugh! This needs cleaning-up, man! We'll try to come back to it later.

Thomas precession as a product of two non-commuting Lorentz transformations We will use a general theorem for the first variation of an arbitrary exponential.

$$\delta\left(e^{A}\right)=\int_{0}^{1}e^{sA}\ \delta A\ e^{(1-s)A}\ ds \tag{4}$$

This is valid even if δA does not commute with A. When $[\delta A,A]=0$, it reduces to the abelian $\delta\left(e^{A}\right)=(\delta A)\,e^{A}=e^{A}\,(\delta A)$, of course. Also note that $e^{A+\delta A}-e^{A}=\delta\left(e^{A}\right)+$ terms involving two or more (δA)s.

One proof of (4) is by series expansion of all exponentials. On the LHS we have

$$\delta\left(e^{A}\right)=\delta\sum_{n=0}^{\infty}\frac{1}{n!}\ A^{n}=\sum_{n=1}^{\infty}\frac{1}{n!}\ \delta\left(A^{n}\right)=\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{m=0}^{n-1}A^{n-1-m}\left(\delta A\right)A^{m} \tag{5}$$

while on the RHS we have

$$\begin{aligned}\int_{0}^{1}e^{sA}\ \delta A\ e^{(1-s)A}\ ds &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{m=0}^{\infty}\frac{1}{m!}A^{n}\left(\delta A\right)A^{m}\int_{0}^{1}s^{n}\left(1-s\right)^{m}ds\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{m=0}^{\infty}\frac{1}{m!}A^{n}\left(\delta A\right)A^{m}\ \frac{n!m!}{(n+m+1)!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A^{n}\left(\delta A\right)A^{m}\ \frac{1}{(n+m+1)!}\end{aligned}$$

Now change summation variables in the double sum. $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}=\sum_{j=0}^{\infty}\sum_{m=0}^{j}$ where $j=n+m$. Thus

$$\int_{0}^{1}e^{sA}\ \delta A\ e^{(1-s)A}\ ds=\sum_{j=0}^{\infty}\sum_{m=0}^{j}A^{j-m}\left(\delta A\right)A^{m}\ \frac{1}{(j+1)!}$$

But this is precisely the double sum that appears in (5), upon renaming $j=n-1$. QED.

Note: To evaluate the parameter integral above, we have used the **beta function** with arguments v, w, as defined by the integral

$$B(v,w)=\int_{0}^{1}s^{v-1}\left(1-s\right)^{w-1}ds$$

The beta function is related to the gamma function as follows:

$$B\left(v,w\right)=\frac{\Gamma\left(v\right)\Gamma\left(w\right)}{\Gamma\left(v+w\right)}$$

where

$$\Gamma\left(y\right)=\int_{0}^{\infty}e^{-x}x^{y-1}dx$$

If v and w are integers, then

$$\begin{aligned}\Gamma\left(y\right) &= (y-1)!\\ B\left(v,w\right) &= \frac{(v-1)!\,(w-1)!}{(v+w-1)!}\end{aligned}$$