[hep-th/0609148] C Zachos

A CLASSICAL BOUND ON QUANTUM ENTROPY

$$\boxed{0 \leq S_q \leq \ln\left(\frac{e\sigma^2}{2\hbar}\right)}$$

involving the variance σ^2 in phase space of the **classical** limit distribution of a given **quantum** system.

Intuitively plausible: $\hbar \rightarrow 0$ information forfeiture, augmenting ignorance.

$\rightsquigarrow$ A fortiori, this further bounds the corresponding information-theoretical generalizations of the quantum entropy proposed by Rényi.

Black Hole entropic behavior: collective flow of information in need of robust estimates through gross geometrical and semiclassical features of the system—instead of toilsome detailed accounts of subtler quantum states.

$\rightsquigarrow$ **Combine** upper bound for the entropy of classical continuous distributions (Shannon, 1949) with well-defined classical limit of intricate quantum systems in phase space (Braunss 1994), which tracks the information loss involved in smearing away quantum effects: The quantum entropy of a system is majorized by that of its 'ignorant' classical limit.

• **Illustrate** by the elementary physics paradigm of a thermal bath of oscillator excitations of one degree of freedom: its phase-space representation is a (maximal entropy $\sim$ chaos) Gaussian.

• Extension to arbitrary degrees of freedom and tighter bounds according to the circumstances of physical applications are conceptually straightforward.

SHANNON INFORMATION ENTROPY

For a continuous distribution function $f(x,p)$ in phase space, the **classical** information entropy is

$$S_{cl} = -\int dxdp\ f\ \ln(f).$$

Given a $f(x,p)$, without loss of generality centered at the origin, normalized, $\int dxdpf = 1$, and
with a given variance, $\sigma^2 = \langle x^2 + p^2 \rangle = \int dxdpf(x^2+p^2)$,
$\leadsto$ elementary constrained variation of this $S_{cl}[f]$ w.r.t. f, $\leadsto$
it is maximized by the Gaussian, $f_g = \exp(-(x^2+p^2)/\sigma^2)/\sigma^2\pi$, to
$S_{g\ cl} = 1 + \ln(\pi\sigma^2)$.
• A Gaussian represents maximal disorder and minimal information.
In thermodynamics, least dispersal energy would be available.

$\leadsto$ Shannon's inequality,

$$\boxed{S_{cl} \leq \ln(\pi e \sigma^2)\ ,}$$

an **upper bound on the lack of information**.

• In general, S_{cl} is unbounded above: it diverges for delocalized distributions ($\sigma \to \infty$), containing no information. In contrast to the Boltzmann-Gibbs entropy, it is also unbounded below, given ultralocalized peaked distributions ($\sigma \to 0$), which reflect complete order and information.

BOLTZMANN GIBBS QUANTUM ENTROPY

In quantum mechanics, the sum over all states is given by the standard von Neumann entropy for a density matrix ρ,

$$0 \leq S_q = -\mathrm{Tr}\ \rho\ \ln \rho = -\langle \ln \rho \rangle\ .$$

$\Rrightarrow$ Transcribes in phase space through the Wigner transition map to

$$\boxed{0 \leq S_q = -\int dxdp\ f\ \ln_\star (hf)\ ,}$$

where Groenewold's (1946) $\star$-product,

$$\star \ \equiv \ e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)}\ ,$$

serves to **define** $\star$-functions, such as this $\star$-logarithm,
e.g., through $\star$-power expansions,

$$\ln_\star (hf) \equiv -\sum_{n=1}^{\infty} \frac{(1-hf)^n_\star}{n}\ .$$

By virtue of the convexity below of the function $x \ln x$, Braunss has proven that, for $S_q + \ln h \to S_{cl}$ as the Planck constant $\hbar \to 0$,

$$\boxed{0 \leq S_q \leq S_{cl} - \ln h \ .}$$

• The logarithmic offset term relying on the Planck constant h accounts for the **scale** of the phase-space area element $dxdp$.

This scale, h, should divide $dxdp$ and multiply f, to preserve 'probability', in the Wigner transition map from the density matrix ρ to the Wigner Function f.

• E.g., for a pure state,

$$f(x,p) = \frac{1}{2\pi} \int dy \ \psi^* \left(x - \frac{\hbar}{2} y \right) \ e^{-iyp} \psi \left(x + \frac{\hbar}{2} y \right) \ .$$

The classical limit often entails activity of phase-space variables much larger than $\hbar$; and the scaling down of these variables to scales matched to such activity. Comparing quantum and classical entropies relies on this offset.

⋔ The upper bound in this Braunss inequality reflects the **loss of quantum information** involved in the smearing implicit in the classical limit.

$\leadsto$ Combined with Shannon's bound, this now amounts to

$$\boxed{0 \leq S_q \leq \ln\left(\frac{e\sigma^2}{2\hbar}\right)},$$

i.e., the entropy is bounded above by an expression involving the variance of the corresponding **classical limit distribution function**.

⊛ Readily generalizes to multidimensional phase space, and contexts where more information (e.g., on asymmetric variances) happens to be available, or refinement desired.

• The quantum entropy is recognized as an expansion

$$S_q = \sum_{n=1}^{\infty} \frac{\langle (1-\rho)^n \rangle}{n} = \sum_{n=1}^{\infty} \frac{\langle (1-hf)_\star^n \rangle}{n}.$$

The leading term, $n=1$, $1-\mathrm{Tr}\rho^2 = \langle 1-hf \rangle$, is the **impurity**, often referred to as linear entropy. Like the entropy itself, it **vanishes for a pure state**, for which $\rho^2=\rho$, or, equivalently, $f\star f = f/h$.

$\leadsto$ Each term in the expansion projects out ρ, or $\star hf$, respectively: **pure states saturate the lower bound on S_q**.

RÉNYI ENTROPY

A likewise additive (extensive) generalization of the quantum entropy is the Rényi entropy,

$$R_\alpha = \frac{1}{1-\alpha} \ \ln\langle\rho^{\alpha-1}\rangle = \frac{1}{1-\alpha} \ \ln\left(\int \frac{dxdp}{h}(hf)^\alpha_\star\right) \ .$$

- The limit $\alpha \to 1$ yields $R_1 = S_q$; and the impurity is $1 - \exp(-R_2)$.

For continuous distributions (infinity of components) discussed here, R_0 is divergent.

⁜ For $\alpha \geq 1$, $R_\alpha \geq R_{\alpha+1}$, so $S_q \geq R_\alpha$, and it is also bounded below by 0,

$$S_q \geq R_\alpha \geq R_{\alpha+1} \geq 0 \ .$$

↬ A fortiori, the Rényi entropy is also **bounded by the same bound**.

GAUSSIAN ILLUSTRATION

Consider the Gaussian Wigner Function of **arbitrary** half-variance E,

$$f(x,p,E) = \frac{e^{-\frac{x^2+p^2}{2E}}}{2\pi E} = e^{-\frac{x^2+p^2}{2E} - \ln(2\pi E)}.$$

This happens to be the phase-space Wigner transform of a Maxwell-Boltzmann thermal distribution for a harmonic oscillator, in suitably rescaled units, normalized properly to unity, and with mean energy $E = \langle \frac{x^2+p^2}{2} \rangle$.

Calculation of the entropy of this distribution, is, of course, a freshman physics problem; review its independent phase-space derivation, evaluate S_q directly.

For $E = \hbar/2$, the distribution reduces to just f_0, the Wigner Function for a **pure state** (the ground state of the harmonic oscillator). $\looparrowright$

$$f_0 \star f_0 = \frac{f_0}{h} \ ,$$

$\leadsto$ f_0 is $\star$-orthogonal to each of the terms in the sum, and hence $S_q = 0$, indicating saturation of the maximum possible information content.

For generic width E, the Wigner Function f is not that of a pure state, but **it still happens to always amount to** a $\star$-exponential ($e_\star^a \equiv 1 + a + a\star a/2! + a\star a\star a/3! + \ldots$) as well,

$$hf = e^{-\frac{x^2+p^2}{2E} + \ln(\hbar/E)} = e_\star^{-\frac{\beta}{2\hbar}(x^2+p^2) + \ln(\frac{\hbar}{E}\cosh(\beta/\hbar))},$$

where an "inverse temperature" variable $\beta(E,\hbar)$ is useful to define,

$$\tanh(\beta/2) \equiv \frac{\hbar}{2E} \leq 1 \qquad \Longrightarrow \qquad \beta = \ln\frac{E+\hbar/2}{E-\hbar/2}\ .$$

(Thus the above pure state f_0 corresponds to zero temperature, $\beta = \infty$.)

Since $\star$-functions, by virtue of their $\star$-expansions, obey the **same** functional relations as their non-$\star$ analogs, inverting the $\star$-exponential through the $\star$-logarithm and integrating yields directly the standard thermal physics result:

$$\boxed{S_q(E,\hbar) = \frac{E}{\hbar}\ln\left(\frac{2E+\hbar}{2E-\hbar}\right) + \frac{1}{2}\ln\left((\frac{E}{\hbar})^2 - \frac{1}{4}\right) = \frac{\beta}{2}\coth(\beta/2) - \ln(2\sinh(\beta/2)).}$$

$\looparrowright$ **monotonically nondecreasing** function of E, attaining the lower bound 0 for the pure state $E \to \hbar/2$ ($\beta \to \infty$, zero temperature).

The classical limit, $\hbar \to 0$ ($\beta \to 0$, infinite temperature) thus follows,

$$S_q \to 1 + \ln(E/\hbar) = \ln(\pi e 2E) - \ln h = S_{cl}(E) - \ln h \ .$$

Explicitly seen to bound the expression for all E; saturating it for large $E >> \hbar$, in accordance with Braunss' bound. I.e., the upper bound is **saturated** for Gaussian quantum Wigner functions with $\sigma^2 >> \hbar$.

⋇ The region $E < \hbar/2$, corresponding to ultralocalized spikes excluded by the uncertainty principle, was **not allowed** by the above derivation method, since, in this region, no $\star$-**Gaussian can be found** to represent the Gaussian. (It would amount to complex β and S_q, linked to thermal expectations of the oscillator parity operator.)

$\star$-powers of the Gaussian are also straightforward to take, and thus the Rényi entropies can also be readily computed:

$$R_\alpha = \frac{1}{1-\alpha} \ln \left(\frac{(2\,\sinh(\beta/2))^\alpha}{2\,\sinh(\alpha\beta/2))} \right) = \frac{1}{\alpha-1} \ln \left(\left(\frac{E}{\hbar} + \frac{1}{2}\right)^\alpha - \left(\frac{E}{\hbar} - \frac{1}{2}\right)^\alpha \right).$$

• $\alpha \to 1$ checks with the above $R_1 \to S_q$. Also, in the pure state limit, $E = \hbar/2$, it is evident that $R_\alpha = 0$ checks for all $\alpha \geq 1$.
(For $\alpha > 1$ and the small disallowed values $E < \hbar/2, \quad R_\alpha < 0$.)

R_α is also a nondecreasing function of E; and a nonincreasing function of α. Up to an additive, α-dependent constant, the classical limit is identical to that for the entropy itself,

$$\boxed{R_\alpha \to \frac{\ln \alpha}{\alpha - 1} + \ln(E/\hbar)}\ .$$

in agreement with the classical result.

It may well be that specific αs could provide more detailed or practical measures of complexity in Hawking radiation with sparse information available.

• **If a specific quantum Hamiltonian were actually available** for the system in question (rare), then the classical limit of the entropy of the system would be straightforward $\hookrightarrow$ our inequality would not be that powerful, since the classical entropy itself would be at hand, in general lower than the Shannon bound.
For such simple systems, the upper-bounding classical entropy would result out of the phase-space partition function specified by the corresponding classical hamiltonian (the Weyl symbol of the quantum hamiltonian).

$\leadsto$ Illustrated explicitly by hamiltonians which are positive N-th powers of the oscillator hamiltonian,

$$\boxed{f_{cl} \propto \exp(-\left(\frac{x^2+p^2}{2E}\right)^N).}$$

By standard thermodynamic evaluation, the bounding classical entropy reduces to just the Shannon entropy,

$$\boxed{S_{cl} = \frac{1}{N} + \ln\left(2\pi E\Gamma(1+\frac{1}{N})\right),}$$

lower than the Shannon bound, $1 + \ln\left(\pi E\frac{\Gamma(1+2/N)}{\Gamma(1+1/N)}\right)$.