On Nonself-adjoint Sturm-Liouville Operators with Matrix Potentials

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We consider the differential operator $L_t(Q)$ generated in the space $L_2^m[0,1]$ by the differential expression

$$l(y) = -y^{''}(x) + Q(x)\,y(x) \tag{1}$$

and the quasiperiodic conditions

$$y^{'}(1) = e^{it}y^{'}(0),\; y(1) = e^{it}y(0) \tag{2}$$

for $t \in [0, 2\pi)$, where $L_2^m[0,1]$ is the set of the vector functions $f(x) = (f_1(x), f_2(x), ..., f_m(x))$ with $f_k(x) \in L_2[0,1]$ for $k = 1, 2, ..., m$ and

$Q(x) = \Big(b_{i,j}(x)\Big)$ is a $m \times m$ matrix with the complex-valued summable entries $b_{i,j}(x)$.

The norm $\|.\|$ and inner product $(.,.)$ in $L_2^m[0,1]$ are defined by

$$\|f\| = \left(\int_0^1 |f(x)|^2 \, dx \right)^{\frac{1}{2}}, \ (f,g) = \int_0^1 \langle f(x), g(x) \rangle \, dx,$$

where $|.|$ and $\langle .,. \rangle$ are respectively the norm and the inner product in $\mathbb{C}^m$. These boundary value problems play a fundamental role in the spectral theory of the differential operator L generated in the space $L_2^m(-\infty,\infty)$ by the expression $-y''(x) + Q(x)\,y(x)$ with the periodic coefficient $Q(x)$, since the spectrum of the operator L is the union of the spectra of L_t for $t \in [0, 2\pi)$

It follows easily from the well-known classical investigations that the eigenvalues of the operator $L_t(Q)$ consist of m sequences

$$\{\lambda_{k,1} : k \in \mathbb{Z}\}, \ \{\lambda_{k,2} : k \in \mathbb{Z}\}, ..., \ \{\lambda_{k,m} : k \in \mathbb{Z}\}$$

lying in the $O(|k|^{1-\frac{1}{m}})$ neighborhoods of the eigenvalues $(2k\pi + t)^2$ of the operator $L_t(0)$.

(M. A. Naimark, *Linear Differential Operators,* Frederick Ungar Pobl. Co. New York, 1967.)

We prove that the eigenvalues $\lambda_{k,j}$ of $L_t(Q)$ lie in the $O\left(\frac{\ln|k|}{k}\right)$ neighborhoods of the eigenvalues of the operator $L_t(C)$, where $C = \int\limits_0^1 Q\,(x)\,dx$. For this we consider $L_t(Q)$ as perturbation of $L_t(C)$ by $Q(x) - C$. It means that we take the operator $L_t(C)$ for an unperturbed operator and $Q(x) - C$ for a perturbation.

Note that to obtain the asymptotic formulas of order $O(\frac{1}{k})$ for the eigenvalues $\lambda_{k,j}$ of $L_t(Q)$, by using the classical asymptotic expansions for the solutions of the matrix equation

$$-Y^{''} + Q\,(x)\,Y = \lambda Y,$$

it is required that $Q(x)$ be differentiable.

1. R. Carlson, "Large Eigenvalues and Trace Formulas for Matrix Sturm–Liouville Problems", SIAM Journal on Mathematical Analysis 30 (5), 949-962 (1999).

2. F. G. Maksudov, O. A. Veliev, "Spectral Analysis of Differential Operators with Periodic Matrix Coefficients", Differential Equations 25 (3), 271-277 (1989).

The suggested method in this paper gives the possibility of obtaining the asymptotic formulas of order $O(k^{-1}\ln|k|)$ for the eigenvalues $\lambda_{k,j}$ and the normalized eigenfunctions $\Psi_{k,j}(x)$ of $L_t(Q)$ when the entries $b_{i,j}(x)$ of $Q(x)$ belong to $L_1[0,1]$. Using these asymptotic formulas we prove that if the eigenvalues of the matrix C are simple, then the root functions of the operator $L_t(Q)$ for $t\neq 0,\pi$ form a Riesz basis.

Theorem 1 *The boundary conditions (2) are regular and the eigenvalues of the operator $L_t(Q)$ for $t\neq 0,\pi$ consist of m sequences $\lambda_{k,j}(t)$ (j=1,2,...,m)$satisfying$*

$$\lambda_{k,j}(t)=(2\pi k+t)^2+O\left(k^{1-\frac{1}{m}}\right), \tag{3}$$

where $k=\pm N,\pm(N+1),...,$ and $N\gg 1$.

The formula (3) shows that the eigenvalue $\lambda_{k,j}$ of the operator $L_t(Q)$ is close to the eigenvalue $(2k\pi+t)^2$ of the operator $L_t(0)$. Moreover, for $t \neq 0, \pi$, the eigenvalue $\lambda_{k,j}$ is far from the other eigenvalues $(2n\pi+t)^2$, where $n \neq k$, of $L_t(0)$. Clearly,

$$\varphi_{n,1} = \begin{pmatrix} e^{i(2\pi n+t)x} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \varphi_{n,2} = \begin{pmatrix} 0 \\ e^{i(2\pi n+t)x} \\ \vdots \\ 0 \end{pmatrix}$$

$$, \ldots, \varphi_{n,m} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{i(2\pi n+t)x} \end{pmatrix}$$

are the eigenfunctions of the operator $L_t(0)$ corresponding to the eigenvalue $(2\pi n+t)^2$. The multiplicity of $(2\pi n+t)^2$, for $t \neq 0, \pi$, is m and the corresponding eigenspace is $E_n = span\left\{\varphi_{n,1}, \varphi_{n,2}, \ldots, \varphi_{n,m}\right\}$. The adjoint operator to $L_t(Q)$ is $L_t(Q^*)$, where $Q^*(x)$ is the adjoint matrix to $Q\left(x\right)$. Since the boundary conditions (2) are self-adjoint, the operator $L_t\left(Q\right)$ is self-adjoint if $Q(x)$ is a symmetric matrix. Hence

$L_t^*(0) = L_t(0).$

Now to analyze the operator $L_t(C)$ we introduce the following notations. Suppose the matrix C has p distinct eigenvalues $\mu_1, \mu_2, ..., \mu_p$ with multiplicities $m_1, m_2, ..., m_p$ respectively, where $m_1 + m_2 + ... + m_p = m$. Let $u_{j,1},\ u_{j,2}, ...u_{j,s_j}$ be the eigenvectors corresponding to the eigenvalue μ_j. Denote by $u_{j,s,1},\ u_{j,s,2}, ...u_{j,s,r_{j,s}-1}$ the associated vectors belonging to the eigenvector $u_{j,s}$. Note that $r_{j,s}$ is called the multiplicity of the eigenfunction $u_{j,s}$ and

$$r_{j,1} + r_{j,2} + ... + r_{j,s_j} = m_j. \tag{4}$$

The number r_j defined by $r_j = \max_s r_{j,s}$ is a maximum multiplicity of the eigenfunctions corresponding to the eigenvalue μ_j.

It is not hard to see that $u_{j,s}e^{i(2\pi k+t)x}$ for $s=1,2,...,$ are the eigenfunctions of $L(C)$ corresponding to the eigenvalue $\mu_{k,j}=(2\pi k+t)^2+\mu_j$. Moreover, $u_{j,s,r}e^{i(2\pi k+t)x}$ for $r=1,2,...,$ are the associated functions of $L(C)$ belonging to the eigenfunction $u_{j,s}e^{i(2\pi k+t)x}$.

Here, for notational convenience, the eigenvalues of C, counted with multiplicity, are indexed as $\mu_1,\mu_2,...,\mu_m$. Any normalized eigenvector corresponding to the eigenvalue μ_j is denoted by v_j. The associated vectors belonging to the eigenvector v_j are denoted by $v_{j,1},v_{j,2},...$. In these notations the eigenvalues, eigenfunctions, associated functions of $L_t(C)$ are

$$\mu_{k,j}=(2\pi k+t)^2+\mu_j,$$

$$\Phi_{k,j}(x)=v_je^{i(2\pi k+t)x},\ \Phi_{k,j,s}(x)=v_{j,s}e^{i(2\pi k+t)x}$$

respectively. Similarly, the eigenvalues, eigenfunctions, and associated functions of $L_t^*(C)$ are $\overline{\mu_{k,j}}$,

$\Phi^*_{k,j}(x) = v^*_j e^{i(2\pi k+t)x}$, and $\Phi^*_{k,j,s}(x) = v^*_{j,s} e^{i(2\pi k+t)x}$, where v^*_j and $v^*_{j,s}$ are the eigenvector and associated vector of C^* corresponding to $\overline{\mu_j}$. By definition

$$(L^*(C) - \overline{\mu_{k,j}})\Phi^*_{k,j}(x) = 0, \tag{5}$$

$$(L^*(C) - \overline{\mu_{k,j}})\Phi^*_{k,j,s}(x) = \Phi^*_{k,j,s-1}(x), \tag{6}$$

where $\Phi^*_{k,j,0}(x) = \Phi^*_{k,j}(x)$. Multiplying both sides of

$$L(Q)\Psi_{k,j}(x) = \lambda_{k,j}\Psi_{k,j}(x) \tag{7}$$

by $\Phi^*_{k,j}(x)$, using $L(Q) = L(C) + (Q - C)$ and (5), we get

$$\begin{aligned} & (\lambda_{k,j} - \mu_{k,j})(\Psi_{k,j}(x), \Phi^*_{k,j}(x)) \\ = \ & ((Q(x) - C)\Psi_{k,j}(x), \Phi^*_{k,j}(x)). \end{aligned} \tag{8}$$

Now multiplying (7) by $\Phi^*_{k,j,1}(x)$ and using (6), (8), we obtain

$$\begin{aligned} & (\lambda_{k,j} - \mu_{k,j})^2(\Psi_{k,j}(x), \Phi^*_{k,j,1}(x)) \\ = \ & ((Q(x) - C)\Psi_{k,j}(x), \Phi^*_{k,j}(x)) + \\ & (\lambda_{k,j} - \mu_{k,j})((Q(x) - C)\Psi_{k,j}(x), \Phi^*_{k,j,1}(x)). \end{aligned} \tag{9}$$

In this way one can deduce the formulas

$$(\lambda_{k,j}-\mu_{k,j})^{s+1}(\Psi_{k,j}(x),\Phi^{*}_{k,j,s}(x)) = \tag{10}$$

$$\sum_{p=0}^{s}(\lambda_{k,j}-\mu_{k,j})^{p}((Q(x)-C)\Psi_{k,j}(x),\Phi^{*}_{k,j,p}(x)).$$

To obtain the asymptotic formulas we estimate the terms

$$((Q(x)-C)\Psi_{k,j}(x),\Phi^{*}_{k,j,p}(x)),$$
$$(\Psi_{k,j}(x),\Phi^{*}_{k,j,s}(x)).$$

Lemma 2 *If* $t \neq 0, \pi$, *then*

$$\Big(\Psi_{k,j}(x),(Q^{*}(x)-C^{*})\Phi^{*}_{k,i,p}(x)\Big) = O(\frac{\ln|k|}{k}),$$

where $i = 1, 2, ..., m$ *and* $p = 0, 1, 2, ...$

Lemma 3 *For each eigenfunction* $\Psi_{k,j}(x)$ *of* $L_t(Q)$, *where* $|k| \geq N$, *there exists a root function* $\Phi^{*}_{k,i,s}(x)$ *of* $L_t(C^{*})$ *satisfying*

$$\left|\Big(\Psi_{k,j}(x),\Phi^{*}_{k,i,s}(x)\Big)\right| \geq \frac{1}{2m}.$$

Theorem 4 *If $t \neq 0, \pi$, then:*

(a) *All large eigenvalues of $L_t(Q)$ lie in $O((\frac{\ln|k|}{k})^{\frac{1}{r_j}})$ neighborhood of the eigenvalues $\mu_{k,j} = (2\pi k + t)^2 + \mu_j$ of $L_t(C)$, where $k \in \mathbb{Z}$, $j = 1, 2, ..., m$, the number r_j is a maximum multiplicity of the eigenfunctions corresponding to the eigenvalue μ_j and $r_j = 1$ if the matrix C has no associated vector corresponding to the eigenvalue μ_j.*

(b) *Let μ_j be a simple eigenvalue of the matrix C and $\lambda_{k,j}$ be an eigenvalue of $L_t(Q)$ lying in $\frac{1}{2}a_j$ neighborhood of $\mu_{k,j} = (2\pi k + t)^2 + \mu_j$, where*

$a_j = \min_{i \neq j} \mid \mu_j - \mu_i \mid$. *Then $\lambda_{k,j}$ is the simple eigenvalue of $L_t(Q)$. Moreover, $\lambda_{k,j}$ and the corresponding eigenfunction $\Psi_{k,j}(x)$ satisfy*

$$\lambda_{k,j}(t) = (2\pi k + t)^2 + \mu_j + O(\frac{\ln|k|}{k}), \tag{11}$$

$$\Psi_{k,j}(x) = v_j e^{i(2\pi k + t)x} + O(\frac{\ln|k|}{k}), \tag{12}$$

where v_j *is the eigenvector of* C *corresponding to the eigenvalue* μ_j.

(c) *Suppose that all eigenvalues* $\mu_1, \mu_2, ..., \mu_m$ *of* C *are simple. Then there exists a number* N *such that the all eigenvalues* $\lambda_{k,1}, \lambda_{k,2},...,\lambda_{k,m}$ *of* $L_t(Q)$ *for* $\mid k \mid\geq N$ *are simple and satisfy the asymptotic formula (11). The eigenfunction* $\Psi_{k,j}(x)$ *of* $L(Q)$ *satisfies (12). The root functions of* $L_t(Q)$ *form a Riesz basis in* $L_2^m(0,1)$.