Teardrops and Dyonic Strings

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• Scalar induced compactifications
• Matter coupled gauged $\mathcal{N} = (1, 0)$ 6D SUGRA
• Killing spinor and integrability conditions
• A new dyonic string solution

Earlier dyonic string solutions in 6D:

Duff, Ferrara, Khuri & Rahmfeld, 1995
Guven, Liu, Pope & Sezgin, 2004
Scalar induced compactifications

Considered in the 80’s as alternative to $p$-form induces Freund-Rubin type compactifications.

Motivated by search for chiral compactifications. Unexpected novel features such as absence of massless KK vectors.

Omero-Percacci found identity map solution in Einstein plus (compact) sigma model (1980)

Gell-Mann-Zweibach used $SU(1,1)/U(1)$ sector of Type IIB SUGRA to curl up two spatial dimensions (1984). 1/2 supersymmetric and known as teardrop solution.

More recently:

Supersymmetric solutions of $N=2$ $d=4$ sugra (Huebscher, Meessen & Ortin)

The 6D superswirl (Parameswaran, Tasinato & Zavala). Solution to gauged $N=(1,0)$ SUGRA plus sigma model. Novel, static, supersymmetric, co-did 2 object.
The Gell-Mann-Zweibach Teardrop

\[ R_{MN} = \frac{1}{2} h_{\alpha\beta} \partial_M \phi^\alpha \partial_N \phi^\beta \]

\[ \nabla_M \nabla^M \phi^\alpha + \Gamma_{\beta\gamma}^\alpha \partial_M \phi^\beta \partial_M \phi^\gamma = 0 \]

Metric on \( SU(1, 1)/U(1) \):

\[ h_{\alpha\beta} = \frac{4}{1 - (\phi^1)^2 - (\phi^2)^2} \delta_{\alpha\beta} \]

Let \( x^M = (x_\mu, y^i) \) and consider identity map:

\[ \phi^\alpha = y^i \delta^\alpha_i , \quad i = 1, 2 \]

This gives:

\[ R_{ij} = \frac{1}{2} h_{ij}(x^k) , \quad R_{\mu\nu} = 0 \]

With this satisfied, scalar field equation contains no further information.
If sigma model manifold was compact, then \( g_{ij} = h_{ij} \) would be a solution, as in the Omero & Percacci solution. But here it is not a solution because it gives negative \( R_{ij} \) while the RHS of the equation is manifestly positive.

The following metric is, however, a solution:

\[
 g_{ij} = \frac{4a^2}{1 - (y^1)^2 - (y^2)^2} \delta_{ij}
\]

Size parameter \( a^2 \) undetermined by equation of motion.

The power of the scale factor is determined by the coefficient on the RHS of Einstein equation.

Teardrop with \( U(1) \) rotational symmetry. Noncompact with finite volume.

Solution preserves half supersymmetry.

**Question:**

Sigma model sectors exist in many (un)gauged SUGRA's in \( D \leq 10 \). Do they have classes of solutions that generalize this teardrop?
Chiral, gauged SUGRA in 6D

\[ N = (1, 0) \] supersymmetry multiplets:

- **graviton** \( g_{\mu\nu}, \psi_{\mu+}^A, B_{\mu\nu}^- \)
- **tensor (dilaton)** \( \varphi, \chi^A, B_{\mu\nu}^+ \)
- **hypermatter** \( \phi^{AA'}, \psi_{A'}^A, (A, A' = 1, 2) \)
- **Yang – Mills** \( A_M, \lambda_+^A \)

R-symmetry group is \( Sp(1) \), and either the full R-symmetry or a \( U(1) \) subgroup can be gauged.

**General couplings of such gauged supergravity were constructed long ago** (Nishino and Sezgin, 1984).
• Anomaly free such models are very rare. Only known ones without drones ($U(1)$ not acting on any fermion):

$$E_7 \times E_6 \times U(1)_R \text{ model} \quad (912,1,1)$$

$$E_7 \times G_2 \times U(1)_R \text{ model} \quad (56,14,1)$$

$$F_4 \times Sp(9) \times U(1)_R \text{ model} \quad (52,18,1)$$

• Intricate potentials and remarkable vacua e.g. has no Minkowski$_6$ but natural $4 + 2$ split.

• Promising applications in braneworld scenarios. (Quevedo, Burgess et al)

• String/M theory origin is still mysterious, though some progress made.
The Killing Spinor Method

General form of supersymmetric solutions in 6D, using Killing spinors, have been studied in:

hep-th/0306235: Gutowski, Martelli and Reall,

hep-th/0402055: Cariglia and MacConamhna,

Both of these studies without hypermatter couplings. In our work, we have closed this gap.

I will explain the method below.
The hyperscalars $\phi^\alpha$, $\alpha = 1, \ldots, 4n_H$ parameterize the coset

$$Sp(n_H, 1)/Sp(n_H) \otimes Sp(1)_R$$

Other quaternionic coset spaces $G/H$ are also possible. We gauge the group

$$K \times Sp(1)_R \subset Sp(n_H, 1), \quad K \subseteq Sp(n_H)$$

The group $K$ is taken to be semi-simple, and the $Sp(1)_R$ part of the gauge group can easily be replaced by its $U(1)_R$ subgroup.

**Gauge $C$-functions:**

$$L^{-1}T^I L \equiv C^I = C^{IaA} T_{aA} + \frac{1}{2} C^{IAB} T_{AB} + \frac{1}{2} C^{Iab} T_{ab}$$

**Covariant derivative**

$$P^a_A = (D_\mu \phi^\alpha)V^{aA}_\alpha, \quad D_\mu \phi^\alpha = \partial_\mu \phi^\alpha - A^I_\mu K^{I\alpha}$$

$K^I(\phi)$ are the Killing vectors that generate the $K \times Sp(1)_R$ transformations on $G/H$. 
\[ e^{-1} \mathcal{L}_B = R - \frac{1}{4} (\partial \varphi)^2 - \frac{1}{2} e^{\varphi} G_{\mu \nu \rho} G^{\mu \nu \rho} - \frac{1}{4} e^{\frac{1}{2} \varphi} F_{\mu \nu}^I F^{I \mu \nu} - 2 P_\mu^a P_{\alpha A}^\mu - 4 e^{-\frac{1}{2} \varphi} C_{AB}^I C^{IAB} \]

where \( dG = \frac{1}{2} F^I \wedge F^I \). The local supersymmetry transformations of the fermions:

\[
\begin{align*}
\delta \psi_\mu &= D_\mu \varepsilon + \frac{1}{48} e^{\frac{1}{2} \varphi} G^\nu_{\mu \sigma \rho} \Gamma^{\nu \sigma \rho} \Gamma_\mu \varepsilon \\
\delta \chi &= \frac{1}{4} \left( \Gamma^\mu \partial_\mu \varphi - \frac{1}{6} e^{\frac{1}{2} \varphi} G^-_{\mu \nu \rho} \Gamma^{\mu \nu \rho} \right) \varepsilon \\
\delta \chi_A^I &= -\frac{1}{8} F^I_{\mu \nu} \Gamma^{\mu \nu} \varepsilon_A - e^{-\frac{1}{2} \varphi} C^I_{AB} \varepsilon^B \\
\delta \psi^a &= P_\mu^a \Gamma^\mu \varepsilon_A 
\end{align*}
\]
**Fermionic bilinears:**

Only two nonvanishing fermionic bilinears possible:

\[ \bar{\epsilon}^A \Gamma_\mu \epsilon^B \equiv V_\mu \epsilon^{AB}, \quad \bar{\epsilon}^A \Gamma_{\mu \nu \rho} \epsilon^B \equiv X_{\mu \nu \rho}^r T_{r}^{AB} \]

From the Fierz identity \( \Gamma_{\mu (\alpha \beta \gamma) \delta} = 0 \), it follows that

\[ V_\mu V_\mu = 0 \quad i_V X^r = 0 \]

Introducing the orthonormal basis \( ds^2 = 2 e^+ e^- + e^i e^i \) and identifying \( e^+ = V \), the equation \( i_V X^r = 0 \) implies \( X^r = 2 V \wedge I^r \) where

\[ I^r = \frac{1}{2} I_{ij}^r e^i \wedge e^j \]

is anti-self dual in the metric \( ds^2_4 = e^i e^i \). Fierz identities imply that \( I^r \) is the quaternionic structure. They also give:

\[ V_\mu \Gamma^{\mu} \epsilon = 0 \]

- **Natural emergence of 2+4 split.**
- **In other dimensions** \( V^\mu \) maybe timelike, spacelike or null, leading to solution classification accordingly.
On Killing Spinor Conditions:

- Necessary and sufficient condition for $\delta \psi_{\mu}$:

\[
\nabla_\mu V_\nu = -\frac{1}{2} e^{2\varphi} G^{+\nu}_\mu V^\rho
\]

\[
D_\mu I^r_{ij} = e^{2\varphi} G^{+k}_{\mu[i} I^r_{j]k}
\]

where $D_\mu I^r \equiv \nabla_\mu I^r + \epsilon_{rst} Q^s_{\mu} I^t$.

Thus: $\nabla_{(\mu} V_{\nu)} = 0 \Rightarrow$ null Killing vector.

- From $\delta \lambda = 0$, we learn in particular that

\[
\left(\frac{1}{8} I^r_{ij} \Gamma^{ij}_B \delta^A_B - T^r A B\right) \epsilon^B = 0 \Rightarrow 1/8 SUSY
\]

- From $\delta \psi^a = 0$ we learn in particular that

\[
D_i \phi^\alpha = (I^r)_i^j (J^r)^{\alpha}{}_{\beta} D_j \phi^\beta
\]

Transverse space & sigma model quaternionic structures intertwined here.
Summary

- All the field strengths and the metric are determined to some extent.

- The Killing spinor integrability conditions imply all the field equations except:

\[ R_{++} = T_{++} \]
\[ D_\mu \left( e^{\varphi/2} F_{\mu I}^+ \right) = 0 \]
\[ D_\mu (e^\varphi G^{\mu \nu \rho}) = 0 \]
\[ DF^I = 0 \]
\[ dG = \frac{1}{2} F^I \wedge F^I \]
A New Dyonic String Solution

Define

\[ e^{\varphi^-} = \left( \frac{2a}{3b} \right)^2 h^{1/3}, \quad e^{\varphi^+} = 3\nu \left( \frac{a}{b} \right)^2 h^{1/3} + \nu_0 \]

\[ h = \frac{y^2}{a^2} - 1, \quad a, b, \nu \text{ constants} \]

Our solution is:

\[ ds^2 = e^{-\frac{1}{2}\varphi^+} e^{-\frac{1}{2}\varphi^-} (-dt^2 + dx^2) \]

\[ + e^{\frac{1}{2}\varphi^+} e^{\frac{1}{2}\varphi^-} \left( \frac{b}{y^2} \right)^2 h^{2/3} dy^\alpha dy^\beta \delta_{\alpha\beta} \]

\[ e^\varphi = e^{\varphi^+}/e^{\varphi^-} \]

\[ \phi^\alpha = \frac{ay^\alpha}{y^2} \]

\[ A_r^\alpha = \frac{4}{3y^2} \rho_{\alpha\beta}^r y^\beta, \]

\[ G_{\alpha\beta\gamma} = \frac{8}{27(y^2)^2} \epsilon_{\alpha\beta\gamma\delta} y^\delta \]

\[ G_{+-\alpha} = -\partial_\alpha e^{-\varphi^+}, \]

where \( \rho^r_{\alpha\beta} \) are the 't Hooft matrices.
The form of \( h \) dictates that \( a^2 < y^2 < \infty \), covering outside of a disk of radius \( a \). The hyperscalars map this region into \( H^4 \), i.e. the interior of the disk defined by \( \phi^2 < 1 \).

These scalars are gravitating in the sense that their contribution to the energy momentum tensor does not vanish.

It is possible to apply a coordinate transformation and map the base space into the disc by defining

\[
z^\alpha = \frac{ay^\alpha}{y^2}
\]

The metric then takes the form:

\[
ds^2 = e^{-\frac{1}{2} \varphi^+} e^{-\frac{1}{2} \varphi^-} \, dx^2 + L^2 e^{\frac{1}{2} \varphi^+} e^{\frac{1}{2} \varphi^-} \, \left( dr^2 + r^2 d\Omega_3^2 \right)\]

\[\equiv 2H^{-1}H du dv + H ds^2_B,
\]

where \( r = \sqrt{z^2} \) and \( L = a/b \), and the metric on the base space is as follow:
\[ ds_B^2 = L^2 \left( \frac{1}{r^2} - 1 \right)^{2/3} \left( dr^2 + r^2 d\Omega_3^2 \right) \]

\[ \equiv \frac{(1 - r^2)^{8/3}}{2r^{4/3}} ds_{H_4}^2 \]

where \( ds_{H_4}^2 = 2(dr^2 + r^2 d\Omega_3^2)/(1 - r^2)^2 \) is the metric on \( H_4 \).

This is a four dimensional generalization of the Gell-Mann Zweibach teardrop.

- Noncompact with finite volume and rotational symmetry

- The Ricci curvature scalar flips sign because of the conformal rescaling of sigma model metric.

- Curvature scalar of the base metric diverges at the boundary \( r \to 1 \), but unlike GZ teardrop, also at \( r = 0 \).

  This is not a problem, as we can place a source there to soak up the singularity, as we shall see shortly.
**Dyonic charges and coupling to sources:**
Remarkably, our solution has a fixed magnetic charge:

\[ Q_m = \int_{S^3} G = \frac{8}{27} \text{vol}_{S^3} \]

The electric charge, however, is proportional to the constant parameter \( \nu \):

\[ Q_e = \int_{S^3} *e^\varphi G = 2\nu \text{vol}_{S^3} \]

It is not known how to write down the coupling of a dyonic string with nonvanishing electric magnetic charges to sources. But setting the electric charge to zero, we find that the singularities from

\[ \partial_\alpha \partial^\alpha h = -4\pi^2 \delta(\vec{z}) \]

can be accounted for precisely by the coupling the supergravity fields to the magnetically charged string located at \( r = 0 \) by the following action:

\[ S = -\int d^2\sigma e^{\varphi/2} \sqrt{-\gamma} + \int \tilde{B} \]

where \( \gamma \) is the determinant of the induced world-sheet metric and \( \tilde{B} \) is the 2-form potential whose field strength is dual to \( G \).
The Near Horizon Limit

This is the $r \to 0$ limit and it depends on the electric charge parameter $\nu$.

$\nu \neq 0$ The limit is $AdS_3 \times S^3$ with $1/4$ SUSY

$\nu = 0$ Defining $du = dr/r^{5/6}$, the limit becomes:

$$ds^2 \sim u^2 (-dt^2 + dx^2 + d\Omega_3^2) + du^2$$

Ignoring $x$ and $\Omega_3$ directions, this describes the Rindler wedge which is the near horizon geometry of the Schwarzschild black hole.

Many of the properties of our dyonic string differ significantly from the known dyonic string solutions.
Future Directions

- Full fluctuation analysis?

- Implications of the new dyonic string for the embedding the $6D$ gauged supergravity in string theory?

- General class of tear-drop like solutions in (un)gauged supergravities in all dimensions? Braneworld via tear-drop?

- Implication for the landscape of string theory vacua?