Shift Theorem and Schwinger Mechanism in the Presence of Arbitrary Time-Dependent Background Electric Field

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- Introduction:
  Schwinger Mechanism in the Presence of
  Constant Electric Field: A historical sketch

- Shift Theorem Involving Sum
  of Non-Commuting Operators in Path Integrals

- Schwinger Mechanism in the Presence
  of Arbitrary Time Dependent Electric Field

- Conclusions

Introduction:

- Schwinger Obtained the Following Non-Perturbative Formula for e^+e^- Pair Production From Constant Electric Field via Vacuum Polarization by Using Proper Time Method:

- $\frac{dW}{d^4x} = e^2E^2 \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{n\pi m^2}{|eE|}}$

- The p_T Distribution Can not Be Obtained by using Proper Time Method

- The p_T Distribution Can be Obtained by Direct Path Integration :

- $\frac{dW}{d^4x d^2p_T} = -|eE| \ Log[1 - e^{-\frac{\pi(p_T^2+m^2)}{|eE|}}]$

⋆ However, Schwinger Mechanism or the Calculation of One-loop Effective Action in Physics is so far Restricted to Constant Background Fields

⋆ What Our Work is About:

- We Study, for the first time, Schwinger Mechanism in the Presence of Arbitrary Time Dependent Background Electric Field $E(t)$

- We Encounter Infinite Number of Non-Commuting Terms Between $E(t)$ and $\frac{d}{dt}$ in The Exponential

- All These Terms Were Zero in Schwinger's Original Calculation Because The Constant Field E Commutes with $\frac{d}{dt}$

- We develop a "Shift Theorem" Involving Sum of Non-Commuting Operators Inside Exponential in Path Integrals to Solve This Problem:

- "Shift Theorem" We Found is Given by:(hep-th/0609192)

$$\int_{-\infty}^{+\infty} dx f_1(y) < x|e^{-[(a(y)x+h\frac{d}{dy})^2+b(\frac{d}{dx})+c(y)]}|x > f_2(y)$$
$$= \int_{-\infty}^{+\infty} dx f_1(y) < x - \frac{h}{a(y)}\frac{d}{dy}|e^{-[a^2(y)x^2+b(\frac{d}{dx})+c(y)]}|x - \frac{h}{a(y)}\frac{d}{dy} > f_2(y).$$

- Using This We Find The Following Non-Perturbative Formula For Charged Scalar Production From Arbitrary Time Dependent Electric Field $E(t)$ via Schwinger Mechanism:(hep-th/0611125)

- $\frac{dW}{d^4xd^2p_T} = |eE(t)|\ Log[1 + e^{-\frac{\pi(p_T^2+m^2)}{|eE(t)|}}]$

- Now I Will Present the Derivation of These Results

- General Formulation of Schwinger Mechanism in Scalar QED

- $< 0|0 >^A = \frac{\int [d\phi^\dagger][d\phi] e^{i\int d^4x \phi^\dagger M[A]\phi}}{\int [d\phi^\dagger][d\phi] e^{i\int d^4x \phi^\dagger M[0]\phi}} = Det^{-1}[M[A]]/Det^{-1}[M[0]] = e^{iS^{(1)}}$

- where for scalar field theory $M[A] = (\hat{p} - eA)^2 - m^2$
  and The one loop effective action is given by

- $S^{(1)} = iTrln[(\hat{p} - eA)^2 - m^2] - iTrln[\hat{p}^2 - m^2]$
  $= -i\int_0^\infty \frac{ds}{s} \int d^4x < x|[e^{-is[(\hat{p}-eA)^2-m^2-i\epsilon]} - e^{-is(\hat{p}^2-m^2-i\epsilon)}]|x >$

- We assume that the time dependent electric field E(t) is along the z-axis. We choose the Axial gauge $A_3 = 0$ so that $A_0 = -E(t)z$

- $S^{(1)} = -i \int_0^\infty \frac{ds}{s} \int_{-\infty}^{+\infty} dt < t| \int_{-\infty}^{+\infty} dx < x| \int_{-\infty}^{+\infty} dy < y| \int_{-\infty}^{+\infty} dz < z|$
$e^{-is[(\hat{p}_0+eE(t)z)^2-\hat{p}_z^2-\hat{p}_T^2-m^2-i\epsilon]} - e^{-is(\hat{p}^2-m^2-i\epsilon)}$
$|z > |y > |x > |t >$

- Inserting complete set of $|p_T >$ states $(\int d^2p_T|p_T >< p_T| = 1)$:

- $S^{(1)} = \frac{-i}{(2\pi)^2} \int_0^\infty \frac{ds}{s} \int d^2x_T \int d^2p_T e^{is(p_T^2+m^2+i\epsilon)} [\int_{-\infty}^{+\infty} dt < t|$
$\int_{-\infty}^{+\infty} dz < z| e^{-is[(-i\frac{d}{dt}+eE(t)z)^2-\hat{p}_z^2]} |z > |t > - \int dt \int dz \frac{1}{4\pi s}]$

- This Involves Infinite Number of Non-Commuting Terms Between $E(t)$ and $\frac{d}{dt}$ in The Exponential

- Hence This is a Much Difficult Problem Than the Schwinger's Constant Electric Field E Problem

- "Shift Theorem" Involving Sum of Non-Commuting Operators in the Exponential inside Path Integrals

- $x \pm \frac{h}{a(y)}\frac{d}{dy} = e^{\pm\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}} x e^{\mp\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}$

- $a(y)x + h\frac{d}{dy} = e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}[e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}} a(y) e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}} x] e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}$

- $(a(y)x + h\frac{d}{dy})^2 =$
  $e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}[(e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}} a(y) e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}} x)^2] e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}$

- $(a(y)x + h\frac{d}{dy})^2 + b(\frac{d}{dx}) + c(y) = e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}$
  $[(e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}} a(y) e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}} x)^2 + b(\frac{d}{dx}) + e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}} c(y) e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}]$
  $e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}$

- $e^{-[(a(y)x+h\frac{d}{dy})^2+b(\frac{d}{dx})+c(y)]} = e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}$
  $e^{-[(e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}a(y)e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}x)^2+b(\frac{d}{dx})+e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}c(y)e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}]}$
  $e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}$

- Hence
  $I(y) = \int_{-\infty}^{+\infty} dx f_1(y) < x|e^{-[(a(y)x+h\frac{d}{dy})^2+b(\frac{d}{dx})\ +c(y)]}|x > \ f_2(y)$
  $= \int_{-\infty}^{+\infty} dx f_1(y) \ < x|e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}$
  $e^{-[(e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}a(y)e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}x)^2+b(\frac{d}{dx})+e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}\ c(y)e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}]}$
  $e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}\ |x > f_2(y)$

- Next we change the x integration variable to x' where
  $x = x' - \frac{h}{a(y)}\frac{d}{dy}$

- The limits for x' remain $\pm\infty$. Under this change of integration variable one also has $dx = dx'$. Hence

- $I(y) = \int_{-\infty}^{+\infty} dx f_1(y) < x - \frac{h}{a(y)}\frac{d}{dy}|e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}$
$e^{-[(e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}a(y)e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}(x-\frac{h}{a(y)}\frac{d}{dy}))^2+b(\frac{d}{dx})+e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}c(y)e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}]}$
$e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}|x - \frac{h}{a(y)}\frac{d}{dy} > \ f_2(y)$

- Using, $$x - \frac{h}{a(y)}\frac{d}{dy} = e^{-\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}} x e^{\frac{h}{a(y)}\frac{d}{dy}\frac{d}{dx}}$$

- $I(y) = \int_{-\infty}^{+\infty} dx f_1(y) < x|e^{-[(a(y)x+h\frac{d}{dy})^2+b(\frac{d}{dx})+c(y)]}|x > f_2(y)$
$= \int_{-\infty}^{+\infty} dx f_1(y) < x - \frac{h}{a(y)}\frac{d}{dy}|e^{-[a^2(y)x^2+b(\frac{d}{dx})+c(y)]}$
$|x - \frac{h}{a(y)}\frac{d}{dy} > \ f_2(y)$

- Explicit Verification of "Shift Theorem" For a Special Case:

- W=$\int_{-\infty}^{+\infty} dx\ f_1(y) < x|\ e^{-(a(y)x+h\frac{d}{dy})^2}\ |x >\ f_2(y)$
$= \int_{-\infty}^{+\infty} dx\ f_1(y)\ < x - \frac{h}{a(y)}\frac{d}{dy}|\ e^{-a^2(y)x^2}\ |x - \frac{h}{a(y)}\frac{d}{dy} >\ f_2(y)$

- The Left Hand Side Gives
$\frac{dW}{dp} = \frac{1}{2\pi}\ \int_{-\infty}^{+\infty} dx\ f_1(y)\ e^{-(a(y)x+h\frac{d}{dy})^2}\ f_2(y)$

- The Right Hand Side Gives
$\frac{dW}{dp} = \frac{1}{2\pi}\ \int_{-\infty}^{+\infty} dx\ f_1(y)\ e^{-a^2(y)x^2}\ f_2(y)$

- Hence as a result of the shift theorem we find
$\int_{-\infty}^{+\infty} dx\ e^{-(a(y)x+h\frac{d}{dy})^2}\ f(y) = \int_{-\infty}^{+\infty} dx\ e^{-a^2(y)x^2} f(y) = \frac{\sqrt{\pi}}{a(y)} f(y)$

- We Will Explicitly Verify This by Direct Expansion of the Exponential and Then Integrating Over x From $-\infty$ to $+\infty$

- For Sum of Two Non-Commuting Operators A and B:
  $e^{-(A+B)} = e^{-A}[1 + \sum_{n=1}^{\infty}(-1)^n \prod_{i=1}^{n}[\int_0^{x_{i-1}} dx_i e^{x_i A} B e^{-x_i A}]]$

- $e^{-(a(y)x+h\frac{d}{dy})^2} f(y) =$
  $e^{-A}[1 - \int_0^1 dx_1 e^{x_1 A} B e^{-x_1 A} + \int_0^1 dx_1 e^{x_1 A} B e^{-x_1 A} \int_0^{x_1} dx_2 e^{x_2 A} B e^{-x_2 A}$
  $- \int_0^1 dx_1 e^{x_1 A} B e^{-x_1 A} \int_0^{x_1} dx_2 e^{x_2 A} B e^{-x_2 A} \int_0^{x_2} dx_3 e^{x_3 A} B e^{-x_3 A} + ..] f(y)$

- Where
  $(a(y)x + h\frac{d}{dy})^2 \;=\; A + B,$
  $A \;= a^2(y)x^2,$
  $B = 2xa(y)h\frac{d}{dy} + xh\frac{da(y)}{dy} + h^2\frac{d^2}{dy^2}$

- Integrating over x from $-\infty$ to $+\infty$ We Write
  $\int_{-\infty}^{+\infty} dx e^{-(a(y)x+h\frac{d}{dy})^2} f(y) =$
  $\int_{-\infty}^{+\infty} dx e^{-a^2(y)x^2}[1 - I_1 + I_2 - I_3 + I_4 + ..]$

- Where
$I_n = \frac{a(y)}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx e^{-a^2(y)x^2} [\prod_{i=1}^{n} [\int_0^{x_{i-1}} dx_i e^{x_i a^2(y) x^2} B e^{-x_i a^2(y) x^2}]] f(y)$

- After Performing the $x_i's$ and x Integration We Find
$-I_1 = -\frac{f[y]a'[y]^2}{2a[y]^2} + \frac{a'[y]f'[y]+\frac{1}{2}f[y]a''[y]}{a[y]} - f''[y]$

- $I_2 = \frac{5f[y]a'[y]^4}{24a[y]^4} + \frac{-a'[y]^3 f'[y] - f[y]a'[y]^2 a''[y]}{a[y]^3} + \frac{1}{a[y]^2}(\frac{1}{2}f[y]a'[y]^2 + 2a'[y]f'[y]a''[y] + \frac{3}{8}f[y]a''[y]^2 + \frac{4}{3}a'[y]^2 f''[y] + \frac{7}{12}f[y]a'[y]a^{(3)}[y]) + \frac{1}{a[y]}(-a'[y]f'[y] - \frac{1}{2}f[y]a''[y] - \frac{7}{6}a''[y]f''[y] - \frac{2}{3}f'[y]a^{(3)}[y] - a'[y]f^{(3)}[y] - \frac{1}{6}f[y]a^{(4)}[y]) + (f''[y] + \frac{1}{2}f^{(4)}[y])$

- Two Powers of $\frac{d}{dy}$ Are Contained Only in I_1 and I_2

- Adding $(-I_1)$ and I_2 We Find

- $I_2 - I_1 = \frac{5f[y]a'[y]^4}{24a[y]^4} - \frac{a'[y]^2(a'[y]f'[y]+f[y]a''[y])}{a[y]^3} + \frac{1}{a[y]^2}(\frac{3}{8}f[y]a''[y]^2 + \frac{4}{3}a'[y]^2f''[y] + a'[y](2f'[y]a''[y] + \frac{7}{12}f[y]a^{(3)}[y])) + \frac{1}{6a[y]}(-7a''[y]f''[y] - 4f'[y]a^{(3)}[y] - 6a'[y]f^{(3)}[y] - f[y]a^{(4)}[y]) + \frac{1}{2}f^{(4)}[y]$

- Hence we Find $I_2 - I_1$ is Independent of two Powers of $\frac{d}{dy}$

- Similarly, Four Powers of $\frac{d}{dy}$ Are Contained Only in I_2, I_3 and I_4. We Find For $-I_1 + I_2 - I_3 + I_4$:

- $-I_1+I_2-I_3+I_4=$

$$-\frac{30203(f[y]a'[y]^8)}{1920a[y]^8}+\frac{a'[y]^6(2675a'[y]f'[y]+8734f[y]a''[y])}{120a[y]^7}-$$
$$\frac{1}{6720a[y]^6}(a'[y]^4(80a'[y](6699f'[y]a''[y]+1118a'[y]f''[y])+$$
$$f[y](134120a'[y]^2+659013a''[y]^2+$$
$$177750a'[y]a^{(3)}[y])))+\frac{1}{336a[y]^5}(a'[y]^2(12997f[y]a''[y]^3+$$
$$6a'[y]a''[y](4076f'[y]a''[y]+$$
$$2839f[y]a^{(3)}[y])+126a'[y]^3(62f'[y]+9f^{(3)}[y])+$$
$$3a'[y]^2(3617a''[y]f''[y]+2556f'[y]a^{(3)}[y]+$$
$$f[y](6034a''[y]+663a^{(4)}[y]))))+\frac{1}{13440a[y]^4}$$
$$(-28261f[y]a''[y]^4-4a'[y]a''[y]^2(46496f'[y]a''[y]+$$
$$53199f[y]a^{(3)}[y])-4a'[y]^2(16a''[y]$$
$$(2913a''[y]f''[y]+4988f'[y]a^{(3)}[y])+f[y](102158a''[y]^2+$$
$$16201a^{(3)}[y]^2+22440a''[y]a^{(4)}[y]))-$$
$$168a'[y]^4(852f''[y]-53f^{(4)}[y])-8a'[y]^3(20188f[y]a^{(3)}[y]+$$

$$6708f''[y]a^{(3)}[y] + 2520a''[y]f^{(3)}[y] +$$
$$240f'[y](287a''[y] + 18a^{(4)}[y]) + 987f[y]a^{(5)}[y])) +$$
$$\frac{1}{1680a[y]^3}(a''[y](a''[y](19a''[y]f''[y] +$$
$$2464f'[y]a^{(3)}[y]) + 2f[y](1428a''[y]^2 + 775a^{(3)}[y]^2 +$$
$$471a''[y]a^{(4)}[y])) + 2a'[y](-a''[y](2335f''[y]a^{(3)}[y] + 2877a''[y]f^{(3)}[y]) +$$
$$f'[y](6888a''[y]^2 + 206a^{(3)}[y]^2 - 328a''[y]a^{(4)}[y]) +$$
$$f[y](368a^{(3)}[y]a^{(4)}[y] + a''[y](4053a^{(3)}[y] +$$
$$33a^{(5)}[y]))) + 1680a'[y]^3(f^{(3)}[y] - f^{(5)}[y]) +$$
$$a'[y]^2(-6132a^{(3)}[y]f^{(3)}[y] + 658f[y]a^{(4)}[y] -$$
$$3432f''[y]a^{(4)}[y] + 84a''[y](122f''[y] - 83f^{(4)}[y]) +$$
$$6f'[y](868a^{(3)}[y] - 187a^{(5)}[y]) - 165f[y]a^{(6)}[y]))$$
$$+ \frac{1}{240a[y]^2}(99f[y]a^{(3)}[y]^2 + 244f''[y]a^{(3)}[y]^2 +$$
$$184f'[y]a^{(3)}[y]a^{(4)}[y] + 16f[y]a^{(4)}[y]^2 +$$
$$a''[y]^2(560f''[y] + 373f^{(4)}[y]) + 30f[y]a^{(3)}[y]a^{(5)}[y] +$$

$$a''[y](720a^{(3)}[y]f^{(3)}[y] + 186f[y]a^{(4)}[y] +$$
$$434f''[y]a^{(4)}[y] + 10f'[y](68a^{(3)}[y] + 15a^{(5)}[y]) +$$
$$160a'[y](7f^{(3)}[y] + 3f^{(5)}[y]) + 23f[y]a^{(6)}[y]) +$$
$$120a'[y]^2(3f^{(4)}[y] + f^{(6)}[y]) + 2a'[y](258f^{(3)}[y]a^{(4)}[y] +$$
$$307a^{(3)}[y]f^{(4)}[y] + 64f[y]a^{(5)}[y] +$$
$$6f''[y](90a^{(3)}[y] + 23a^{(5)}[y]) + f'[y](288a^{(4)}[y] + 43a^{(6)}[y]) + 6f[y]a^{(7)}[y]))$$
$$\frac{1}{120a[y]}(-320a^{(3)}[y]f^{(3)}[y] - 234f''[y]a^{(4)}[y] -$$
$$260a''[y]f^{(4)}[y] - 86a^{(4)}[y]f^{(4)}[y]$$
$$- 94f'[y]a^{(5)}[y] - 62f^{(3)}[y]a^{(5)}[y] - 120a'[y]f^{(5)}[y]$$
$$- 80a^{(3)}[y]f^{(5)}[y] - 16f[y]a^{(6)}[y] -$$
$$29f''[y]a^{(6)}[y] - 50a''[y]f^{(6)}[y] -$$
$$8f'[y]a^{(7)}[y] - 20a'[y]f^{(7)}[y] - f[y]a^{(8)}[y])$$
$$+ \frac{1}{24}(8f^{(6)}[y] + f^{(8)}[y])$$

- Which is Independent of Four Powers of $\frac{d}{dy}$

- Hence We Find $-I_1 + I_2 - I_3 + I_4$ is Independent of Four Powers of $\frac{d}{dy}$.

- This Pattern Continues and the Coefficients of Each n Power of $\frac{d}{dy}$ (where n =1,2,3,4...∞) Vanish.

- Hence We Explicitly Verify That:
  $-I_1 + I_2 - I_3 + I_4 - I_5 + I_6 + ... = 0$

- As a Result of Shift Theorem We Find That

- $\int_{-\infty}^{+\infty} dx e^{-(a(y)x+h\frac{d}{dy})^2} f(y) =$
  $\int_{-\infty}^{+\infty} dx e^{-a^2(y)x^2} [1 - I_1 + I_2 - I_3 + I_4 + ..]$
  $= \int_{-\infty}^{+\infty} dx e^{-a^2(y)x^2} f(y) = \frac{\sqrt{\pi}}{f(y)} a(y)$

- Using Shift Theorem We Find For the Schwinger Mechanism:

$$S^{(1)} = \frac{-i}{(2\pi)^2}\int_0^\infty \frac{ds}{s}\int d^2x_T \int d^2p_T e^{is(p_T^2+m^2+i\epsilon)}[\int_{-\infty}^{+\infty} dt < t|$$
$$\int_{-\infty}^{+\infty} dz < z|e^{-is[(-i\frac{d}{dt}+eE(t)z)^2-\hat{p}_z^2]}|z > |t > - \int dt \int dz \frac{1}{4\pi s}]$$
$$= \frac{i}{(2\pi)^2}\int_0^\infty \frac{ds}{s}\int d^2x_T \int d^2p_T e^{is(p_T^2+m^2+i\epsilon)}$$
$$[\int dt \int dz \frac{1}{4\pi s} - \int_{-\infty}^{+\infty} dt < t| \int_{-\infty}^{+\infty} dz$$
$$< z + \frac{i}{E(t)}\frac{d}{dt}|e^{-is[(e^2E^2(t)z^2-\hat{p}_z^2]}|z + \frac{i}{E(t)}\frac{d}{dt} > |t >]$$

- Inserting Complete Set of $|p_z >$ States We Find:

$$S^{(1)} = \frac{i}{(2\pi)^2}\int_0^\infty \frac{ds}{s}\int d^2x_T \int d^2p_T e^{is(p_T^2+m^2+i\epsilon)}$$
$$[\int dt \int dz \ \frac{1}{4\pi s} - \int_{-\infty}^{+\infty} dt < t| \int dp_z \int dq_z \int_{-\infty}^{+\infty} dz < z + \frac{i}{E(t)}\frac{d}{dt}|p_z >$$
$$< p_z|e^{is[(-e^2E^2(t)z^2+\hat{p}_z^2]}|q_z >< q_z|z + \frac{i}{E(t)}\frac{d}{dt} > |t >]$$
$$= \frac{i}{(2\pi)^2}\int_0^\infty \frac{ds}{s}\int d^2x_T \int d^2p_T e^{is(p_T^2+m^2+i\epsilon)}[\int dt \int dz \ \frac{1}{4\pi s} - F]$$

- Where

$$F = \frac{1}{(2\pi)} \int_{-\infty}^{+\infty} dt < t| \int dp_z \int dq_z \int_{-\infty}^{+\infty} dz\ e^{izp_z} e^{-\frac{1}{E(t)}\frac{d}{dt}p_z}$$
$$< p_z|e^{is[(-e^2E^2(t)z^2+\hat{p}_z^2]}|q_z > e^{\frac{1}{E(t)}\frac{d}{dt}q_z} e^{-izq_z}|t >$$

- Inserting Complete Sets of $|z>$ and $|p_0>$ States:

$$F = \frac{1}{(2\pi)} \int_{-\infty}^{+\infty} dt \int dp_0 \int dp_0' \int dp_0'' \int dp_0''' \int dp_z \int dq_z \int dz_1 \int dz_2$$
$$\int_{-\infty}^{+\infty} dz < t|p_0 >< p_0|e^{izp_z} e^{-\frac{1}{E(t)}\frac{d}{dt}p_z}|p_0' >< p_0'| < p_z|z_1 >$$
$$< z_1|e^{is[(-e^2E^2(t)z^2+\hat{p}_z^2]}|z_2 >< z_2|q_z > |p_0'' >< p_0''|e^{\frac{1}{E(t)}\frac{d}{dt}q_z} e^{-izq_z}|p_0''' >$$
$$< p_0'''|t >$$
$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} dt \int dp_0 \int dp_0' \int dp_0'' \int dp_0''' \int dp_z \int dq_z \int dz_1 \int dz_2 \int_{-\infty}^{+\infty} dz$$
$$e^{itp_0} e^{izp_z} < p_0|e^{-\frac{1}{E(t)}\frac{d}{dt}p_z}|p_0' > e^{-iz_1p_z} < p_0'| < z_1|e^{is[(-e^2E^2(t)z^2+\hat{p}_z^2]}|z_2 >$$
$$|p_0'' > e^{iz_2q_z} < p_0''|e^{\frac{1}{E(t)}\frac{d}{dt}q_z}|p_0''' > e^{-izq_z} e^{-itp_0'''}$$

- Inserting complete set of harmonic oscillator states ($\sum_n |n_t><n_t| = 1$)
$F = \frac{1}{(2\pi)^3} \sum_n \int_{-\infty}^{+\infty} dt \int dp_0 \int dp'_0 \int dp''_0 \int dp'''_0 \int dp_z \int dq_z \int dz_1$
$\int dz_2 \int_{-\infty}^{+\infty} dz e^{itp_0} e^{iz(p_z-q_z)}$
$< p_0|e^{-\frac{1}{E(t)}\frac{d}{dt}p_z}|p'_0 > e^{-iz_1p_z} < p'_0| < z_1|n_t >$
$e^{-seE(t)(2n+1)} < n_t|z_2 > |p''_0 > e^{iz_2q_z} < p''_0|e^{\frac{1}{E(t)}\frac{d}{dt}q_z}|p'''_0 > e^{-itp'''_0}$

- Performing the z Integration From $-\infty$ to $+\infty$,
$\int_{-\infty}^{+\infty} dz e^{iz(p_z-q_z)} = 2\pi\delta(p_z - q_z)$,
and Inserting Complete Set of $|t>$ States We Find

- $F = \frac{1}{(2\pi)^3} \sum_n \int_{-\infty}^{+\infty} dt \int dt_1 \int dp_0 \int dp'_0 \int dp''_0 \int dp'''_0 \int dp_z \int dz_1 \int dz_2$
$e^{itp_0} < p_0|e^{-\frac{1}{E(t)}\frac{d}{dt}p_z}|p'_0 > e^{-iz_1p_z} e^{-it_1p'_0} < z_1|n_{t_1} > e^{-seE(t_1)(2n+1)}$
$< n_{t_1}|z_2 > e^{it_1p''_0} e^{iz_2p_z} < p''_0|e^{\frac{1}{E(t)}\frac{d}{dt}p_z}|p'''_0 > e^{-itp'''_0}$

- It Can Be Seen That $< p_0|e^{-\frac{1}{E(t)}\frac{d}{dt}p_z}|p'_0 >$ and $< p''_0|e^{\frac{1}{E(t)}\frac{d}{dt}p_z}|p'''_0 >$ are independent of t and $\frac{d}{dt}$

- $< p_0|f(t)\frac{d}{dt}|p'_0 >= \int dt' \int dt'' \int dp''''_0 < p_0|t' >< t'|f(t)|t'' > < t''|p''''_0 >< p''''_0|\frac{d}{dt}|p'_0 >= \int dt' \int dt'' \int dp''''_0 \; e^{-it'p_0}\; \delta(t'-t'')f(t'')e^{it''p''''_0}\; ip'_0\; \delta(p''''_0 - p'_0) = ip'_0 \int dt'\; e^{-it'(p_0-p'_0)} f(t')$

- Hence t Integration Becomes Simple, $\int_{-\infty}^{+\infty} dt e^{it(p_0-p'''_0)} = 2\pi\delta(p_0 - p'''_0)$

- We Find
$$F = \frac{1}{(2\pi)^2}\sum_n \int dt_1 \int dp_0 \int dp'_0 \int dp''_0 \int dp_z \int dz_1 \int dz_2$$
$$< p''_0|e^{\frac{1}{E(t)}\frac{d}{dt}p_z}|p_0 >< p_0|e^{-\frac{1}{E(t)}\frac{d}{dt}p_z}|p'_0 >$$
$$e^{-iz_1p_z}e^{-it_1p'_0} < z_1|n_{t_1} > e^{-seE(t_1)(2n+1)} < n_{t_1}|z_2 > e^{it_1p''_0}e^{iz_2p_z}$$

- Since $\int dp_0 |p_0><p_0| = 1$ We Find

- $F = \frac{1}{(2\pi)^2} \sum_n \int dt_1 \int dp'_0 \int dp_z \int dz_1 \int dz_2 e^{-iz_1 p_z} < z_1 | n_{t_1} >$
  $e^{-seE(t_1)(2n+1)} < n_{t_1} | z_2 > e^{iz_2 p_z}$
  $= \frac{1}{(2\pi)} \sum_n \int dt \int dp'_0 \int dz | < z | n_t > |^2 \, e^{-seE(t)(2n+1)}$
  $= \frac{1}{(2\pi)} \sum_n \int dt \int dp'_0 e^{-seE(t)(2n+1)}$

- Using the Lorentz force equation
  $dp_\mu = eF_{\mu\nu} dx^\nu, \quad dp_0 = eE(t) dz,$ We Find

- $F = \frac{1}{(4\pi)} \int dt \int dz \frac{eE(t)}{sinh(seE(t))}$

- Hence the Effective Action becomes

- $S^{(1)} = \frac{i}{16\pi^3} \int_0^\infty \frac{ds}{s} \int d^4x \int d^2 p_T e^{is(p_T^2 + m^2 + i\epsilon)} [\frac{1}{s} - \frac{eE(t)}{sinh(seE(t))}]$

- Using The Series Expansion
  $\frac{1}{sinhx} = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^2 n^2 + x^2}$

- And Performing the s-Contour Integration Around the Poles
  $s = \frac{in\pi}{|eE(t)|}$, We Find For the Imaginary Part of The Effective Action

- $W = 2ImS^{(1)} = \frac{1}{8\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int d^4x \int d^2p_T |eE(t)| e^{-n\pi \frac{p_T^2 + m^2}{|eE(t)|}}$

- $\frac{dW}{d^4x d^2p_T} = \frac{|eE(t)|}{8\pi^3} Log[1 + e^{-\pi \frac{p_T^2 + m^2}{|eE(t)|}}]$

- This is The Final Result for Non-Perturbative Charged Scalar Production From Arbitrary Time Dependent Electric Field $E(t)$ Via Schwinger Mechanism

Conclusions

- We Have Obtained a "Shift Theorem" Involving Sum of Non-Commuting Operators in the Exponential inside Path Integrals

- Using the "Shift Theorem" We have Studied, For the First Time, Schwinger Mechanism in the Presence of Arbitrary Time Dependent Background Electric Field

- The Non-Perturbative Result of Pair Production From Arbitrary Time Dependent Electric Field via Schwinger Mechanism is Found to be Independent of all the Time Derivatives $\frac{d^n E(t)}{dt^n}$