Boundary Liouville Theory

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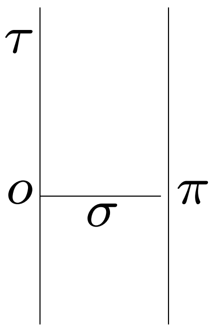
Liouville equation

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)\varphi(\tau,\sigma) + 4m^2\, e^{2\varphi(\tau,\sigma)} = 0$$



Strip: $\sigma \in (0,\pi),\ \tau \in \mathbb{R}$; space-time coordinates

$x = \tau + \sigma,\ \ \bar{x} = \tau - \sigma;$ chiral coordinates

Exponential field $V = e^{-\varphi}$

Liouville equation is equivalent to

$$V\,\partial^2_{x\bar{x}} V - \partial_x V\,\partial_{\bar{x}} V = m^2\ , \qquad V > 0\ .$$

Conformal transformations of the strip: $x \mapsto \xi(x)\ ,\ \bar{x} \mapsto \xi(\bar{x})$;

$$\xi'(x) > 0\ , \qquad \xi(x+2\pi) = \xi(x) + 2\pi\ ; \qquad \xi(x) \in \widetilde{\mathrm{Diff}}_+(S^1)\ .$$

The boundaries $\sigma = 0$ and $\sigma = \pi$ are invariant.

Conformal symmetry of Liouville theory:

the space of solutions is invariant under

$$V(x,\bar{x}) \mapsto \frac{1}{\sqrt{\xi'(x)\xi'(\bar{x})}}\, V(\xi(x),\xi(\bar{x}))$$

Conformally invariant boundary conditions

$a.$ Dirichlet $\quad V|_{\sigma=0} = 0 = V|_{\sigma=\pi}$ (ZZ)

$\varphi \to +\infty$ at the boundaries.

$b.$ Neumann $\quad \partial_\sigma V|_{\sigma=0} = -2m\,l\ , \qquad \partial_\sigma V|_{\sigma=\pi} = 2m\,r$ (FZZT)

(Gervais and Neveu)

$$\partial_\sigma \varphi = 2m\,l\ e^{\varphi}\ |_{\sigma=0}, \qquad \partial_\sigma \varphi = -2m\,r\ e^{\varphi}\ |_{\sigma=\pi}$$

Energy-momentum tensor

$$T = \frac{\partial^2_{xx} V(x,\bar{x})}{V(x,\bar{x})} \;, \qquad \bar{T} = \frac{\partial^2_{\bar{x}\bar{x}} V(x,\bar{x})}{V(x,\bar{x})}$$

Chirality $\quad \partial_{\bar{x}} T = 0 = \partial_x \bar{T}; \quad T(\tau+\sigma), \quad \bar{T}(\tau-\sigma)$

Energy density

$$\mathcal{E} = T + \bar{T} = \frac{1}{2}\,(\partial_\tau \varphi)^2 + \frac{1}{2}\,(\partial_\sigma \varphi)^2 + 2m^2\, e^{2\varphi} - \partial^2_{\sigma\sigma}\varphi$$

Energy flow

$$\mathcal{P} = T - \bar{T} = \partial_\tau \varphi \; \partial_\sigma \varphi - \partial^2_{\tau\sigma}\varphi$$

Vanishing flow at the boundaries

$$T(\tau) = \bar{T}(\tau) \qquad \text{and} \qquad T(\tau + 2\pi) = T(\tau)$$

The general form of V

$$V(x,\bar{x}) = m\ [a\,\psi(\bar{x})\psi(x) + b\,\psi(\bar{x})\chi(x) + c\,\chi(\bar{x})\psi(x) + d\,\chi(\bar{x})\chi(x)]$$

Solutions of Hill's equation

$$\psi''(x) = T(x)\,\psi(x), \qquad \chi''(x) = T(x)\,\chi(x)$$

with the unit Wronskian

$$\psi(x)\chi'(x) - \psi'(x)\chi(x) = 1$$

and the $SL(2,\mathbb{R})$ condition $ad - bc = 1$.

Notation

$$\mathbf{\Psi} = \begin{pmatrix} \psi \\ \chi \end{pmatrix} , \qquad \mathbf{\Psi}^T = (\psi \quad \chi) , \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Matrix form: $V(x, \bar{x}) = m\, \mathbf{\Psi}^T(\bar{x})\, A\, \mathbf{\Psi}(x)$.

Monodromy property $\mathbf{\Psi}(x + 2\pi) = M\, \mathbf{\Psi}(x)$, with $M \in SL(2, \mathbb{R})$.

$SL(2, \mathbb{R})$ maps: $\mathbf{\Psi} \mapsto S\, \mathbf{\Psi}$, $\qquad M \mapsto S\, M\, S^{-1}$.

Hyperbolic M_h , parabolic M_p and elliptic M_e monodromies

$$|\mathrm{Tr}\, M_h| > 2 , \quad |\mathrm{Tr}\, M_p| = 2 , \quad |\mathrm{Tr}\, M_e| < 2 .$$

We first construct V-fields for constant $T(x) = \mathcal{T}$
and then consider their conformal orbits.

This construction covers all regular Liouville fields on the strip.

For $\mathcal{T} > 0$ (hyperbolic monodromy)

$$\psi(x) = \cosh(\sqrt{\mathcal{T}}\, x) \ , \qquad \chi(x) = \frac{1}{\sqrt{\mathcal{T}}} \sinh(\sqrt{\mathcal{T}}\, x) \ .$$

For $\mathcal{T} = 0$ (parabolic monodromy) $\psi(x) = 1, \quad \chi(x) = x$.

For $\mathcal{T} < 0$ (elliptic monodromy)

$$\psi(x) = \cos(\sqrt{|\mathcal{T}|}\, x) \ , \qquad \chi(x) = \frac{1}{\sqrt{|\mathcal{T}|}} \sin(\sqrt{|\mathcal{T}|}\, x) \ , \qquad .$$

FZZT field

$$V = \frac{2m}{p \sinh \pi p} \left[u \cosh p(\tau - \tau_0) + l \cosh p(\sigma - \pi) + r \cosh p\sigma \right] \ .$$

τ_0 is an arbitrary constant, $p = 2\sqrt{\mathcal{T}}$ and

$$u = \sqrt{l^2 + r^2 + 2l\, r \cosh \pi p + \sinh^2 \pi p} \ .$$

The admissible values of the boundary parameters are $l \geq -1$, $r \geq -1$.

For a given (l, r), there are the following restrictions on the variable $\mathcal{T}$:

- If $l + r \geq 0$, then $\mathcal{T} > 0$;
- If $l + r < 0$, then $\mathcal{T} \geq \mathcal{T}_*$, where $\mathcal{T}_* = -\frac{1}{4}\theta_*^2$, with

$$\cos \pi\theta_* = -lr + \sqrt{(1 - l^2)(1 - r^2)} \ .$$

We have all three monodromies if (l, r) is inside the triangle ABC and only the hyperbolic monodromy if (l, r) is outside of it.

In the first case there is a smooth continuation from the hyperbolic to the elliptic sector.

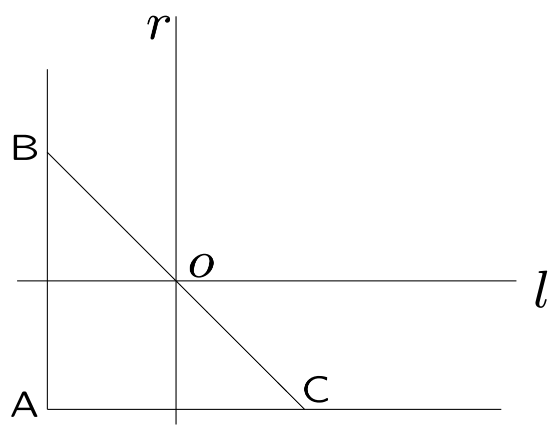


Fig. 1

For $l = r = -1$, $\theta_* = 1$ and V-field is the ZZ vacuum $V = 2m\sin\sigma$.

Thus, $l = r = -1$ the FZZT case contains the ZZ case.

The conformal orbits of the Liouville fields

$$V = \frac{m\,(\xi'(x)\xi'(\bar{x}))^{-\frac{1}{2}}}{\sqrt{\mathcal{T}}\,\sinh 2\pi\sqrt{\mathcal{T}}}\Big(u\,\cosh\sqrt{\mathcal{T}}[\xi(x)+\xi(\bar{x})] +$$

$$l\,\cosh\sqrt{\mathcal{T}}[\xi(x)-\xi(\bar{x})-2\pi] + r\,\cosh\sqrt{\mathcal{T}}[\xi(x)-\xi(\bar{x})]\Big)\ ,$$

where $\xi(x)$ is the group parameter on the orbits and the parameter τ_0 is absorbed by the zero mode of $\xi(x)$.

The energy-momentum tensor provides the co-adjoint orbit of $T(x) = \mathcal{T}$

$$T(x) = \mathcal{T}\,\xi'^{\,2}(x) + \left(\frac{\xi''(x)}{2\xi'(x)}\right)^2 - \left(\frac{\xi''(x)}{2\xi'(x)}\right)'\ .$$

The action of FZZT

$$S = \int d\tau \int_0^\pi d\sigma \frac{1}{2}\left[(\partial_\tau \varphi)^2 - (\partial_\sigma \varphi)^2 - 4m^2 e^{2\varphi}\right] - 2m \int d\tau \left[l\, e^{\varphi(\tau,0)} + r\, e^{\varphi(\tau,\pi)}\right] .$$

It is equivalent to the canonical action

$$S = \int d\tau \int_0^\pi d\sigma \left[\pi\, \dot{\varphi} - \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_\sigma \varphi)^2 + 2m^2\, e^{2\varphi} - \partial^2_{\sigma\sigma}\varphi\right)\right]$$

with $\dot{\phi} = \pi$. The corresponding canonical 2-form is

$$\omega_0 = \int_0^\pi d\sigma\ \delta\pi(\tau,\sigma) \wedge \delta\varphi(\tau,\sigma).$$

The canonical 2-form calculated in the variables $(\mathcal{T}, \xi)$ reads

$$\omega = \delta\mathcal{T} \wedge \int_0^{2\pi} dx\, \xi'(x)\delta\xi(x) + \mathcal{T} \int_0^{2\pi} dx\, \delta\xi'(x) \wedge \delta\xi(x) + \frac{1}{4} \int_0^{2\pi} dx\, \frac{\delta\xi''(x) \wedge \delta\xi'(x)}{\xi'^{\,2}(x)} .$$

The symplectic form provides the following Poisson brackets

$$\{\mathcal{T}, \xi(x)\} = \frac{1}{2\pi} , \qquad \{\xi(x), \xi(y)\} = \frac{1}{4\mathcal{T}} \left(\frac{\sinh 2\sqrt{\mathcal{T}}\, \lambda(x,y)}{\sinh 2\pi\sqrt{\mathcal{T}}} - \frac{\lambda(x,y)}{\pi} \right) ,$$

where $\lambda(x,y) = \xi(x) - \xi(y) - \pi\epsilon(x-y)$.

$\epsilon(x)$ is the stair-step function: $\epsilon'(x) = 2\delta(x)$.

Using the Fourier mode expansion

$$\xi(x) = x + \sum_{n\in\mathbb{Z}} \xi_n \, e^{-inx} \, ,$$

we find that $2\pi\xi_0$ is the canonical conjugated to $\mathcal{T}$

$$\{\mathcal{T}, \, 2\pi\xi_0\} = 1 \ .$$

For $\mathcal{T} < 0$ the variable $\alpha = 2\sqrt{-\mathcal{T}}\,\xi_0$ is cyclic: $(\alpha \ \sim \ \alpha + 2\pi)$, since the exponentials in V become oscillating.

The canonical conjugated to α is $2\pi\sqrt{-\mathcal{T}}$.

This leads to the semi-classical quantization of s:

$$\mathcal{T}_n = -\left(\theta_*(l,r) - \frac{\hbar n}{2\pi}\right)^2 \ .$$

Transformation to free-field variables (for $\mathcal{T} > 0$, $p = 2\sqrt{\mathcal{T}}$):

$$\phi(x) = \frac{p\xi(x)}{2} + \frac{1}{2}\log\xi'(x) - \frac{1}{2}\log\frac{m\,u}{p\,\sinh\pi p}$$

The inverse map

$$\xi(x) = \frac{1}{p}\log\frac{m\,u\,A_p(x)}{2\,\sinh^2\pi p}\ ,$$

where $A_p(x)$ is given by

$$A_p(x) = \int_0^{2\pi} dy e^{2\phi(x+y)-\pi p}\ .$$

The free-field form of ω follows from the direct computation

$$\omega = \int_0^{2\pi} dx\, \delta\phi'(x) \wedge \delta\phi(x) + \delta p \wedge \delta\phi(0)$$

The Fourier mode expansion

$$\phi(x) = \frac{q}{2\pi} + \frac{px}{2} + \frac{i}{\sqrt{4\pi}} \sum_{n\neq 0} \frac{a_n}{n}\, e^{-inx} \ ,$$

provides the canonical brackets

$$\{p, q\} = 1, \qquad \{a_n, a_m\} = in\delta_{n+m,0} \ .$$

The energy-momentum takes also a free-field form

with a linear 'improved' term

$$T(x) = \phi'^2(x) - \phi''(x).$$

Free-field parametrization of V-field

$$V = e^{-[\phi(x)+\phi(\bar{x})]}\left[1 + m\, b_p\, A_p(x) + m, c_p\, A_p(\bar{x}) + m^2\, d_p\; A_p(x) A_p(\bar{x})\right] \; ,$$

where

$$b_p = \frac{l\, e^{-\pi p} + r}{2 \sinh^2 \pi p} \, , \qquad c_p = \frac{l\, e^{\pi p} + r}{2 \sinh^2 \pi p} \, , \qquad d_p = \frac{u(l, r; p)}{4 \sinh^4 \pi p} .$$

The field $\Phi = \phi(x) + \phi(\bar{x})$ is the full free-field on the strip.

It satisfies the Neumann boundary conditions $\partial_\sigma \Phi|_{\sigma=0} = 0 = \partial_\sigma \Phi|_{\sigma=\pi}$

and has the following mode expansion

$$\Phi(\tau, \sigma) = \frac{q}{\pi} + p\tau + \frac{i}{\sqrt{\pi}} \sum_{n \neq 0} \frac{a_n}{n}\, e^{-in\tau} \cos n\sigma \; .$$

Since $p > 0$, $A_p(x)$ and $A_p(\bar{x})$ vanish for $\tau \to -\infty$.

Therefore $\Phi(\tau, \sigma)$ is the *in*-field:

$$\varphi(\tau, \sigma) \to \Phi(\tau, \sigma) \ , \qquad \text{for} \quad \tau \to -\infty \ .$$

The chiral *out*-field is introduced similarly, replacing p by $-p$.

It has a similar mode expansion

$$\phi_{out}(x) = \frac{\tilde{q}}{2\pi} - \frac{px}{2} + \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{\tilde{a}_n}{n} e^{-inx} \ .$$

The relation between *in* and *out* fields

$$\phi_{out}(x) = \phi(x) - \log \frac{m\, u\, A_p(x)}{2 \sinh^2 \pi p}$$

defines a canonical map from the modes $(p, q; a_n)$ to $(\tilde{q}, -p; \tilde{a}_n)$.

Canonical commutation relations

$$[q,p] = i\hbar, \qquad [a_m, a_n^*] = \hbar\, m\, \delta_{mn} \qquad (m > 0, n > 0)\ .$$

Free-field decomposition:

$$\phi(x) = \phi_0(x) + \phi_+(x) + \phi_-(x),$$

$$\phi_0(x) = \frac{q}{2\pi} + \frac{px}{2}, \quad \phi_+(x) = -\frac{i}{\sqrt{4\pi}} \sum_{n>0} \frac{a_n^*}{n}\, e^{inx}, \quad \phi_-(x) = \frac{i}{\sqrt{4\pi}} \sum_{n>0} \frac{a_n}{n}\, e^{-inx}$$

The chiral commutator

$$[\phi(x), \phi(y)] = -\frac{i\hbar}{4}\, \epsilon(x-y)$$

Normal ordered exponential

$$\psi(x) \equiv :e^{-\phi(x)}: = e^{-\phi_+(x)}\; e^{-\phi_-(x)}\; e^{-\phi_0(x)}$$

Screening charge

$$A_p(x) = \int_0^{2\pi} dz\, e^{2\phi_+(x+z)}\; e^{2\phi_-(x+z)}\; e^{2\phi_0(x+z)-\pi p}$$

$$\chi(x) = \psi(x)\, A_p(x)$$

Exchange relations

$$\psi(x)A_p(y) = e^{(i\hbar/2)\,\epsilon(x-y)}\,\frac{\sinh \pi p}{\sinh\left(\pi p + \frac{i\hbar}{2}\right)}\,A_p(y)\,\psi(x) +$$

$$i\sin(\hbar/2)\;\frac{e^{-\pi p \epsilon(x-y)}}{\sinh\left(\pi p + \frac{i\hbar}{2}\right)}\,A_p(x)\,\psi(x)$$

In particular: $\psi(x)\,A_p(x) = A_p(x)\,\psi(x)$.

$$\psi(x)\,\psi(y) = e^{-(i\hbar/4)\,\epsilon(x-y)}\,\psi(y)\,\psi(x)$$

$$\chi(x)\,\chi(y) = e^{-(i\hbar/4)\,\epsilon(x-y)}\,\chi(y)\,\chi(x)$$

$$\psi(x)\,\chi(y) = f_p(x-y)\,\chi(y)\,\psi(x) + g_p(x-y)\,\psi(y)\,\chi(x)$$

where

$$f_p(x-y) = \frac{\sinh\left(\pi p - \frac{i\hbar}{2}\right)}{\sinh \pi p}\; e^{(i\hbar/4)\,\epsilon(x-y)}$$

$$g_p(x-y) = i\sin(\hbar/2)\;\frac{e^{-\pi p\,\epsilon(x-y)}}{\sinh \pi p} e^{(i\hbar/4)\,\epsilon(x-y)}$$

$$V(x,\bar{x}) = [\psi(\bar{x})\,\psi(x) + B_p\,\psi(\bar{x})\,\chi(x) + C_p\,\chi(\bar{x})\,\psi(x) + D_p\,\chi(\bar{x})\,\chi(x)]e^{-i\hbar/8}$$

The locality condition

$$[V(\sigma,-\sigma),\, V(\sigma',-\sigma')] = 0$$

defines the coefficients B_p, C_p and D_p.

$$B_p = m_b\,\frac{l_b\,e^{-(\pi p - i\hbar/2)} + r_b}{2\sinh\pi p\ \sinh(\pi p - i\hbar/2)} \qquad C_p = m_b\,\frac{l_b\,e^{(\pi p + i\hbar/2)} + r_b}{\sinh 2\pi p\ \sinh(\pi p + i\hbar/2)}$$

$$D_p = \frac{m_b^2}{4\sinh\pi p\ \sinh(\pi p + i\hbar)}\left(1 + \frac{l_b^2 + r_b^2 + 2 l_b r_b \cosh(\pi p + i\hbar/2)}{\sinh^2(\pi p + i\hbar/2)}\right)$$

Asymptotic fields

$$e^{-\varphi_{in}} = \psi(\bar{x})\,\psi(x)\,e^{-i\hbar/8}\ , \qquad e^{-\varphi_{out}} = D_p\,\chi(\bar{x})\,\chi(x)\,e^{-i\hbar/8}\ .$$

S-matrix

$$e^{-\varphi_{in}}\,\widehat{S} = \widehat{S}\,e^{-\varphi_{out}}$$

The reflection amplitude

$$\widehat{S}\,|p,0\rangle = R_p|-p,0\rangle$$

The periodic case

$$R_P \sim \frac{\Gamma(iP/b)}{\Gamma(-iP/b)}\,\frac{\Gamma(ib\,P)}{\Gamma(-ib\,P)}\ , \qquad b^2 = \frac{\hbar}{2\pi}$$

(rescale: $P = p/b$).

Equation for R_P in BLT

$$R_P = D_{P-2ib}\, I_P\, R_{P-2ib}\ , \qquad \text{with}$$

$$I_P = \frac{4\pi^2\,\Gamma^2(1+b^2)}{\Gamma(1+b^2+ib\,P)\,\Gamma(1-ib\,P)\,\Gamma(1+2b^2+ib\,P)\,\Gamma(1-b^2-ib\,P)}`.$$

Reflection amplitude for BLT

$$R_P \sim \frac{\Gamma_b(+iP)}{\Gamma_b(-iP)}\,\frac{S_b(\beta^*+is_l+s_r)\,S_b(\beta^*-is_l-s_r)}{S_b(\beta+is_l-s_r)\ \ S_b(\beta-is_l+s_r)}$$

Γ_b is the double-gamma function, $S_b(z) = \Gamma_b(z)/\Gamma_b(Q-z)$;

$\cos(\pi b s_l) = l, \quad \cos(\pi b s_r) = r, \quad Q + iP = 2\beta, \quad Q = b + 1/b.$

Zeros of R_P define the discrete spectrum.

Outlook

- Mechanical model (Morse potential) $e^{2\phi} - \lambda\, e^{\phi}$
- Construction of V_{-n}
- Calculation of correlation functions
- Boundary operators
- S-matrix
- Geometric quantization