The Lambert Oscillator

Notes by T Curtright, January 2007.

Consider the one degree of freedom Hamiltonian

\[
H = \frac{1}{2} p^2 + V(q)
\]

with a nontrivial potential having a “brick wall” at \( q = -1 \), but otherwise non-singular.

\[
V(q) = q - \ln (1 + q) = \frac{1}{2} q^2 - \frac{1}{3} q^3 + \frac{1}{4} q^4 - \frac{1}{5} q^5 + O(q^6)
\]

We plot the potential using Maple plot routines.

The green line represents a trajectory range for \( E = 1 \), whose turning points are given by \( q_- = -1 - \text{LambertW} (-1/e^2) = -0.84141 \) and \( q_+ = -1 - \text{LambertW} (-1, -1/e^2) = 2.1462 \), as explained below. That the turning points can be expressed in terms of the Lambert function is one reason we have chosen to refer to this model as the “Lambert oscillator.”

Moreover, the evolution equation for this oscillator may be thought of as a second order variant of the first order equation obeyed by the Lambert function, providing another justification for the name we have chosen. The second-order equation of motion resulting from \( V(q) \) is

\[
\frac{d^2}{dt^2} q(t) = \frac{-q(t)}{1 + q(t)} = -q(t) + q^2(t) - q^3(t) + \cdots
\]

to be compared with the first-order differential equation satisfied by the Lambert function

\[
\frac{d}{ds} W(s) = \frac{W(s)}{1 + W(s)} = W(s) - W^2(s) + W^3(s) - + \cdots
\]
Note that flipping the sign on the RHS of the Lambert differential equation is achieved just by letting $s \to -s$, while letting $t \to -t$ has no effect on the oscillator equation. More interestingly, it is well-known that the first order Lambert equation is equivalent to the functional relation $W(s)e^{W(s)} = e^s$ up to an overall shift of $s$. What is the functional equation equivalent to the second order oscillator equation? I don't know!

 Alternatively, we could consider the flipped potential

$$U(q) = -q - \ln(1 - q) = \frac{1}{2}q^2 + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + O(q^6)$$

which has the “brick wall” at $q = +1$ and the dubious benefit of having all coefficients positive in the series expansion.

**Series solution for $q(t)$** We first seek a solution without singularities (poles or cuts). So take the time series

$$q(t) = t^\lambda \sum_{n=0}^\infty \frac{q_n}{n!} t^n \quad \text{with} \quad \lambda = 0$$

and require that it satisfy the differential equation written as $(1 + q(t)) \frac{d^2}{dt^2}q(t) = -q(t)$. Well then

$$\begin{align*}
(1 + q(t)) \frac{d^2}{dt^2}q(t) &= \left(1 + \sum_{k=0}^\infty \frac{q_k}{k!} t^k\right) \sum_{l=0}^\infty \frac{q_{l+2}}{l!} t^l \\
-q(t) &= -\sum_{n=0}^\infty \frac{q_n}{n!} t^n
\end{align*}$$
so equating terms of order \(t^n\) gives

\[
\frac{1}{n!} a_{n+2} + \sum_{k=0}^{n} \frac{1}{k! (n-k)!} q_k q_{n-k+2} = -\frac{1}{n!} a_n \quad \Rightarrow \quad \frac{1}{n!} (1 + q_0) a_{n+2} = -\frac{1}{n!} a_n - \sum_{k=1}^{n} \frac{1}{k! (n-k)!} q_k q_{n-k+2}
\]

For example,

\[
t^0 \quad : \quad (1 + q_0) q_2 = -q_0 \quad \Rightarrow \quad q_2 = \frac{-q_0}{1 + q_0}
\]

\[
t^1 \quad : \quad (1 + q_0) q_3 = -q_1 - q_1 q_2 \quad \Rightarrow \quad (1 + q_0) q_3 = -q_1 (1 + q_2) \quad \Rightarrow \quad q_3 = \frac{-q_1}{(1 + q_0)^2}
\]

\[
t^2 \quad : \quad \frac{1}{n!} (1 + q_0) a_{n+2} = -\frac{1}{n!} a_n - \sum_{k=1}^{n} \frac{1}{k! (n-k)!} q_k q_{n-k+2} \bigg|_{n=2} \quad \Rightarrow \quad \frac{1}{2} (1 + q_0) a_4 = \frac{1}{2} q_2 - q_1 q_3 - \frac{1}{2} q_2^2 \quad \Rightarrow \quad q_4 = \frac{q_0 + 2q_1^2}{(1 + q_0)^3}
\]

etc. All coefficients \(q_n\) for \(n > 1\) are given in terms of \(q_0\) and \(q_1\). Moreover, setting \(q_n = a_n / (1 + q_0)^{n-1}\) in

\[
(1 + q_0) q_{n+2} = -q_n - \sum_{k=1}^{n} \frac{n!}{k! (n-k)!} q_k q_{n-k+2}
\]

with \(a_1 = q_1\) and \(a_2 = -q_0\) in particular, then gives

\[
a_{n+2} = -(1 + q_0) a_n - \sum_{k=1}^{n} \frac{n!}{k! (n-k)!} a_k a_{n-k+2} \quad \text{for} \quad n \geq 1
\]

\[
a_{n+2} = -(1 + q_0) a_n - n a_1 a_{n+1} - \sum_{k=2}^{n} \frac{n!}{k! (n-k)!} a_k a_{n-k+2} \quad \text{for} \quad n \geq 2
\]

So, given \(q_0, a_1 = q_1, a_2 = -q_0,\) and \(a_3 = -q_1,\) all \(a_{n>3}\) are determined by this last recursion relation.

**Exercise 1:** Use Mathematica or Maple to evaluate several (say, the first 20) of these coefficients to see if a closed-form solution is obvious.

**Turning points** For a fixed, constant \(H = E\), turning points for the oscillator occur when \(p = 0\), i.e.

\[
E = q - \ln (1 + q) , \quad (1 + q) e^{-q} = e^{-E} , \quad -(1 + q) e^{-(1+q)} = -e^{-1-E}
\]

That is to say

\[
1 + q = -W \left( -e^{-E}/e \right) , \quad q_\pm = 1 - W_\mp \left( -e^{-E}/e \right)
\]

The right turning point \(q_+ \geq 0\) occurs when the lower branch \(W_- \leq -1\) of the Lambert function is used, and the left turning point \(q_- \leq 0\) occurs when the upper branch \(W_+ \geq 1\) is used. Note \(E \geq 0\) implies \(-1/e \leq -e^{-E}/e < 0\). As mentioned earlier, the fact that the turning points of the oscillator are given by the Lambert function is a compelling reason to refer to the model as the Lambert oscillator, in my opinion.
A plot, from pre-programmed Maple definitions of the function, is

\[
\text{LambertW}(x)
\]

Upper branch \(W_+\) (in red) and lower branch \(W_-\) (in blue) of the Lambert function for real \(x\).

Note that \(W_+(z) \equiv \text{LambertW}(z) \equiv \text{LambertW}(0, z)\) while \(W_-(z) \equiv \text{LambertW}(-1, z)\) in Maple’s notation.

Typical solutions are given numerically by Runge-Kutta methods (see Ch 13 in MW, or Press, et al.).

\[
f' = g
\]
\[
g' = \frac{x}{1 + x}, \text{ Functions defined: } f, g
\]
\[
f(0) = 1
\]
\[
g(0) = 0
\]
\[
u' = v
\]
\[
v' = \frac{u}{1 + u}, \text{ Functions defined: } u, v
\]
\[
u(0) = 2
\]
\[
v(0) = 0
\]
\[
j' = k
\]
\[
k' = \frac{j}{1 + j}, \text{ Functions defined: } j, k
\]
\[
j(0) = 0.5
\]
\[
k(0) = 0
\]

I have chosen to give different names to the functions, for different initial data, just for convenience. We plot these numerical solutions.
Correspondingly, the parametric phase space plot is

Exercise 2: Check these numerics.
The time integral \( t(q) \) For a given energy \( E \)

\[
p = dq/dt = \pm \sqrt{2\left(E - q + \ln(1 + q)\right)}
\]

which integrates as (take upper sign, with \( q(0) = 0 \) and \( p(0) = +\sqrt{E} \))

\[
t = \int_0^q \frac{dq}{\sqrt{2\left(E - q + \ln(1 + q)\right)}} = \int_0^{\ln(1+q)} \frac{e^r \, dr}{\sqrt{2\left(E + 1 + r - e^r\right)}}
\]

after changing variables to \( r = \ln(1 + q) \), \( q = e^r - 1 \), so that \( q = 0 \) corresponds to \( r = 0 \). Upon noting for fixed \( E \)

\[
2d\left(\sqrt{E + 1 + r - e^r}\right) = \frac{dr}{\sqrt{E + 1 + r - e^r}} - \frac{e^r \, dr}{\sqrt{E + 1 + r - e^r}}
\]

the time integral may also be rewritten

\[
\int \frac{e^r \, dr}{\sqrt{E + 1 + r - e^r}} = -2\sqrt{E + 1 + r - e^r} + \int \frac{1}{\sqrt{E + 1 + r - e^r}} \, dr
\]

\[
= -2\sqrt{E + 1 + r - e^r} + 2 \frac{d}{dE} \int \sqrt{E + 1 + r - e^r} \, dr
\]

The first of these relations is intriguing. It relates the time integral for the Lambert model to the time integral for a linearly shifted Liouville model with potential

\[
V_{\text{shifted Liouville}}(r) = e^r - r - 1
\]
Alternately, had we omitted the constant 1 from the shifted Liouville potential, then energy $E$ for the Lambert model would identify with $E + 1$ for the shifted Liouville $e^r - r$.

The Lambert and Liouville times are not the same, in general, differing by the additive factor $-2\sqrt{E + 1 + r - e^r}$. This is to be expected since the change of variable $q \mapsto r$ does not convert the Lambert kinetic term into the usual Liouville kinetic term, or vice versa. However, when evaluated at turning points, where $\sqrt{E + 1 + r - e^r} = 0$, we get a direct equality between the periods for the two potential models. See your lecture notes!

So, the basic integral to evaluate is $\int \sqrt{a + r - e^r} \, dr$. This can always be done numerically, if not in terms of well-known functions. Here we just define the integral

$$J(r; E) \equiv \int_0^r \sqrt{E + 1 + \rho - e^\rho} \, d\rho$$

and its companion derivative

$$I(r; E) \equiv 2 \frac{d}{dE} J(r; E) = \int_0^r \frac{1}{\sqrt{E + 1 + \rho - e^\rho}} \, d\rho$$

**Exercise 3:** Name that function! i.e. For a given $E$ look up either of these in a table of integrals and reduce them to a “known function” of $r$.

(NB I do not know the answer in this case.)

If the linear term under the radical were dropped, the integrals would be elementary: $\int_0^r \sqrt{E + 1 - e^\rho} \, d\rho = 2\sqrt{E + 1 - e^r} - 2\sqrt{E + 1} \arctanh \frac{\sqrt{E + 1} - e^r}{\sqrt{E + 1}}$; $\int_0^r \frac{1}{\sqrt{E + 1 + \rho - e^\rho}} \, d\rho = \frac{2}{\sqrt{E + 1}} \left( - \arctanh \frac{\sqrt{E + 1} - e^r}{\sqrt{E + 1}} + \arctanh \frac{\sqrt{E + 1}}{\sqrt{E + 1}} \right)$. But alas, with the linear terms, the integrals are not elementary. Here we are content to note various expansions. For small $r$

$$J(r; E) \equiv \int_0^r \sqrt{E + 1 + \rho - e^\rho} \, d\rho = \sqrt{E} r + \frac{1}{3} \sqrt{E} \left( - \frac{1}{4E} \right) r^3 + \frac{1}{4} \sqrt{E} \left( - \frac{1}{12E} \right) r^4$$

$$+ \frac{1}{5} \sqrt{E} \left( \frac{1}{48E} - \frac{1}{32E^2} \right) r^5 + \frac{1}{6} \sqrt{E} \left( - \frac{1}{240E} - \frac{1}{48E^2} \right) r^6$$

$$+ \frac{1}{7} \sqrt{E} \left( \frac{1}{1440E} - \frac{5}{576E^2} - \frac{1}{128E^3} \right) r^7 + \frac{1}{8} \sqrt{E} \left( - \frac{1}{10080E} - \frac{1}{128E^3} - \frac{1}{360E^2} \right) r^8$$

$$+ \frac{1}{9} \sqrt{E} \left( - \frac{1}{80640E} - \frac{17}{23040E^2} - \frac{7}{1536E^3} - \frac{5}{2048E^4} \right) r^9 + O(r^{10})$$

to be compared to the corresponding integral for the SHO.

$$\int_0^r \sqrt{E - \frac{1}{2} \rho^2} \, d\rho = \sqrt{E} r + \left( - \frac{1}{12\sqrt{E}} \right) r^3 + \left( - \frac{1}{160E^{3/2}} \right) r^5 + \left( - \frac{1}{896E^{5/2}} \right) r^7 + \left( - \frac{5}{18432E^{7/2}} \right) r^9 + O(r^{10})$$

The leading terms in powers of $\frac{1}{E}$ are in agreement between the SHO and shifted Liouville integrals, at least for the odd powers of $r$, as “underbraced” above. Of course, the SHO integral can be done exactly in terms of a well-known function.

$$\int_0^r \sqrt{E - \frac{1}{2} \rho^2} \, d\rho = \frac{1}{4} r \sqrt{4E - 2r^2} + \frac{1}{2} E \sqrt{2} \arctan \frac{r}{\sqrt{2E - r^2}}$$
Also let us compare to the corresponding integral for the symmetric quartic oscillator, as given by
\[
\int_0^r \sqrt{E - \frac{1}{\pi} \rho^2 - \frac{1}{4!} \rho^4} \, d\rho = \sqrt{E}r + \left( -\frac{1}{12\sqrt{E}} \right) r^3 + \left( -\frac{1}{240\sqrt{E}} - \frac{1}{160E^{3/2}} \right) r^5
\]
\[
+ \left( -\frac{1}{1344E^{3/2}} - \frac{1}{896E^{3/2}} \right) r^7 + \left( -\frac{1}{41472E^{3/2}} - \frac{1}{4608E^{3/2}} - \frac{5}{18432E^{3/2}} \right) r^9 + O(r^{11})
\]
This is not so transparent. But again, this integral can be done in terms of known functions.
\[
\int_0^r \sqrt{2E - 12r^2 - r^4} \, d\rho = \frac{r}{6\sqrt{E}} \sqrt{24E - 12r^2 - r^4}
\]
\[
+ \frac{2}{3} \sqrt{2} \frac{\sqrt{E\sqrt{9 + 6E}}}{\sqrt{(-3 + \sqrt{9 + 6E})}} \text{EllipticF} \left( \frac{1}{6\sqrt{E}} \sqrt{3 + \sqrt{9 + 6E}}, \frac{\sqrt{E - 3 - \sqrt{9 + 6E}}}{E} \right)
\]
\[
- \frac{2}{3} \sqrt{2} \frac{\sqrt{E}}{\sqrt{(-3 + \sqrt{9 + 6E})}} \text{EllipticE} \left( \frac{1}{6\sqrt{E}} \sqrt{3 + \sqrt{9 + 6E}}, \frac{\sqrt{E + 3 - \sqrt{9 + 6E}}}{E} \right)
\]

**Exercise 4:** Check this!

For the companion integral
\[
I(r; E) \equiv \int_0^r \frac{1}{\sqrt{E + 1 + \rho - e^\rho}} \, d\rho = \frac{1}{\sqrt{E}} r + \frac{1}{12E^{3/2}} r^3 + \frac{1}{48E^{5/2}} r^4
\]
\[
+ \frac{1}{5\sqrt{E}} \left( \frac{1}{48E} + \frac{3}{32E^2} \right) r^5 + \frac{1}{6\sqrt{E}} \left( \frac{1}{240E} + \frac{1}{16E^2} \right) r^6
\]
\[
+ \frac{1}{7\sqrt{E}} \left( \frac{1}{1440E} + \frac{5}{192E^2} + \frac{5}{128E^3} \right) r^7 + \frac{1}{8\sqrt{E}} \left( \frac{1}{10080E} + \frac{1}{120E^2} + \frac{5}{128E^3} \right) r^8
\]
\[
+ \frac{1}{9\sqrt{E}} \left( \frac{1}{80640E} + \frac{17}{7680E^2} + \frac{35}{1536E^3} + \frac{35}{2048E^4} \right) r^9 + O(r^{10})
\]
and we again compare to the corresponding integral for the SHO.
\[
\int_0^r \frac{1}{\sqrt{E - \frac{1}{2} \rho^2}} \, d\rho = \frac{1}{\sqrt{E}} r + \frac{1}{12E^{3/2}} r^3 + \frac{3}{160E^{5/2}} r^5 + \frac{5}{896E^{7/2}} r^7 + \frac{35}{18432E^{9/2}} r^9 + O(r^{10})
\]
Again, there is a simple exact form for the SHO integral.
\[
\int_0^r \sqrt{E - \frac{1}{2} \rho^2} \, d\rho = \left( \arctan \frac{r}{\sqrt{2E - r^2}} \right) \sqrt{2}
\]
These series have some interesting structure, e.g. the lack of an $r^2$ term and the polynomials in $\frac{1}{E}$, but nothing too useful as far as I can tell.

On the other hand, for large energy, with $\varepsilon = 1/E$,

\[
\frac{1}{\sqrt{E}} J(r; E) \equiv \sqrt{\varepsilon} \int_0^r \frac{1}{\varepsilon + 1 + \rho - e^\rho} d\rho = r + \left( \frac{1}{2} r + \frac{1}{4} r^2 - \frac{1}{2} e^r + \frac{1}{2} \right) \varepsilon + \left( \frac{1}{8} r - \frac{1}{8} r^2 - \frac{1}{16} r^3 + \frac{1}{4} e^r - \frac{1}{16} e^{2r} + \frac{1}{16} \right) \varepsilon^2 \\
+ \left( \frac{1}{16} r^2 - \frac{3}{32} r^2 - \frac{3}{16} e^r + \frac{1}{16} r^3 + \frac{3}{64} e^{2r} + \frac{1}{64} r^4 - \frac{3}{16} e^{3r} + \frac{3}{32} e^{2r} r - \frac{1}{48} e^{3r} + \frac{31}{192} \right) \varepsilon^3 \\
+ \left( -\frac{5}{128} r - \frac{5}{16} e^r - \frac{5}{64} r^2 - \frac{5}{64} r^3 - \frac{5}{128} e^{2r} r - \frac{5}{128} e^{3r} - \frac{1}{144} e^{3r} - \frac{1}{128} e^{2r} r - \frac{5}{512} e^{4r} + \frac{5}{32} e^{2r} r^2 - \frac{5}{96} e^{3r} r + \frac{15}{32} e^{2r} r + \frac{1595}{4608} \right) \varepsilon^4 + O(\varepsilon^5)
\]

\[
\sqrt{E} I(r; E) \equiv \sqrt{\varepsilon} \int_0^r \frac{1}{\varepsilon + 1 + \rho - e^\rho} d\rho = r + \left( -\frac{1}{2} r - \frac{1}{4} r^2 + \frac{1}{2} e^r - \frac{1}{2} \right) \varepsilon + \left( \frac{3}{8} r - \frac{3}{8} r^2 + \frac{1}{8} r^3 - \frac{3}{4} e^r + \frac{3}{16} e^{2r} - \frac{3}{16} \right) \varepsilon^2 \\
+ \left( -\frac{5}{16} r - \frac{15}{32} r^2 + \frac{15}{16} e^r - \frac{5}{16} r^3 - \frac{15}{16} e^{2r} - \frac{5}{64} r^4 + \frac{15}{32} e^{2r} r - \frac{15}{32} e^{3r} + \frac{5}{48} e^{3r} - \frac{155}{192} \right) \varepsilon^3 \\
+ \left( \frac{35}{128} r + \frac{35}{16} e^r + \frac{35}{64} r^2 + \frac{35}{64} r^3 + \frac{105}{128} e^{2r} r + \frac{35}{128} e^{3r} - \frac{35}{144} e^{3r} - \frac{7}{128} e^{2r} r + \frac{35}{512} e^{4r} - \frac{35}{32} e^{3r} r + \frac{105}{128} e^{2r} r^2 - \frac{35}{96} e^{3r} r - \frac{105}{32} e^{2r} r - \frac{11165}{4608} \right) \varepsilon^4 + O(\varepsilon^5)
\]
Perhaps things are more transparent upon expanding in powers of the exponential $e^\rho$. This is achieved by

$$\int \sqrt{b + e^\rho} \, d\rho = 2e^{\frac{1}{2}b} + (-2 - \rho) be^{-\frac{1}{2}b} + \left(6 + 3\rho + \frac{3}{4}\rho^2\right)^2 b^3 e^{-\frac{1}{2}b}$$

$$+ \left(-30 - 15\rho - \frac{15}{4}\rho^2 - \frac{5}{8}\rho^3\right) b^5 e^{-\frac{1}{2}b} + \left(210 + 105\rho + \frac{105}{4}\rho^2 + \frac{35}{8}\rho^3 + \frac{35}{64}\rho^4\right) b^7 e^{-\frac{1}{2}b}$$

$$+ \left(-1890 - 945\rho - \frac{945}{4}\rho^2 - \frac{315}{8}\rho^3 - \frac{315}{64}\rho^4 - \frac{63}{128}\rho^5\right) b^9 e^{-\frac{1}{2}b} + O(b^\rho)$$

Were there just a constant term, instead of a linear one, the integral would be elementary.

$$\int \sqrt{b + e^\rho} \, d\rho = 2\sqrt{b + e^\rho} - 2\sqrt{b} \arctanh \frac{\sqrt{b + e^\rho}}{\sqrt{b}}$$

Then, there is the more enigmatic

$$\int \sqrt{b + ce^\rho} \, d\rho = \left(\frac{2}{3}\rho^2\right) + \left(-\frac{1}{2}i\sqrt{\pi} \text{erf}(i\sqrt{b})\right) c + \left(\frac{1}{4}\frac{e^{2\rho}}{\sqrt{b}} + \frac{1}{3}\frac{e^{4\rho}}{\sqrt{b}} \text{erf}(i\sqrt{b})\right) c^2$$

$$+ \left(-\frac{1}{24}\frac{e^{3\rho}}{\rho} - \frac{1}{4}\frac{e^{5\rho}}{\sqrt{b}} \text{erf}(i\sqrt{b})\right) c^3 + \left(\frac{1}{64}\frac{e^{6\rho}}{\rho^2} + \frac{1}{24}\frac{e^{4\rho}}{\rho} + \frac{1}{3}\frac{e^{4\rho}}{\sqrt{b}} + \frac{2}{3}\frac{e^{4\rho}}{\sqrt{b}} \text{erf}(i\sqrt{b})\right) c^4$$

$$+ \left(\frac{1}{128}\frac{e^{5\rho}}{\rho^2} - \frac{1}{96}\frac{e^{5\rho}}{\rho} + \frac{5}{48}\frac{e^{5\rho}}{\sqrt{b}} - \frac{25}{48}\frac{e^{5\rho}}{\sqrt{b}} \text{erf}(i\sqrt{b})\right) c^5$$

$$+ \left(\frac{7}{1536}\frac{e^{6\rho}}{\rho^2} + \frac{1}{128}\frac{e^{6\rho}}{\rho} + \frac{3}{160}\frac{e^{6\rho}}{\sqrt{b}} + \frac{3}{160}\frac{e^{6\rho}}{\sqrt{b}} \text{erf}(i\sqrt{b})\right) c^6$$

$$+ \left(-\frac{3}{1024}\frac{e^{7\rho}}{\rho^2} - \frac{7}{1536}\frac{e^{7\rho}}{\rho} + \frac{7}{768}\frac{e^{7\rho}}{\sqrt{b}} - \frac{1920}{768}\frac{e^{7\rho}}{\sqrt{b}} \text{erf}(i\sqrt{b})\right) c^7 + O(c^8)$$

where $\int_0^\infty e^{-s^2} \, ds \equiv \frac{1}{2}\sqrt{\pi} \text{erf}(x)$. Again, were there a constant instead of the linear term, this would be easy.

$$\int \sqrt{1 + ce^\rho} \, d\rho = 2\sqrt{1 + ce^\rho} - 2\sqrt{1 + ce^\rho} \arctanh \sqrt{1 + ce^\rho} \arctanh \sqrt{1 + ce^\rho} = \sqrt{1 + ce^\rho} \quad \text{OK}$$

**Perturbation in the linear term** So, all this emphasizes the linear term under the square root is the essential source of difficulty. Therefore, we perturb it in.

$$\int_0^r \sqrt{a + br - e^\rho} \, dr = \sum_{n=0}^\infty \frac{b^n}{n!} \frac{d^n}{da^n} \int_0^r \sqrt{a - e^\rho} \, dr$$

Now, the latter integrals can get pretty hairy. For example,

$$\int \sqrt{a - e^\rho} \, dr = 2\sqrt{a - e^\rho} - 2\sqrt{a} \arctanh \frac{\sqrt{a - e^\rho}}{\sqrt{a}}$$
is easy enough, but then comes the next one
\[ 2 (-2 + r) \sqrt{a - e^r} + 4i \frac{a^{k_1}}{\sqrt{\pi}} \sum_{k_1=0}^{\infty} \frac{a^{k_1}}{2k_1 + 1} \left( 1 - \exp \left( -\frac{1}{2r} (2k_1 + 1) \right) \right) \frac{\Gamma (k_1 + \frac{1}{2})}{\Gamma (k_1 + 1)} \]
\[ -4i \frac{a^{k_1}}{\sqrt{\pi}} \sum_{k_1=0}^{\infty} \frac{a^{k_1}}{(2k_1 + 1)^2} \left( 1 + \frac{1}{2} (-2 - r (2k_1 + 1)) \exp \left( -\frac{1}{2r} (2k_1 + 1) \right) \right) \frac{\Gamma (k_1 + \frac{1}{2})}{\Gamma (k_1 + 1)} \]
\[ = 2 (-2 + r) \sqrt{a - e^r} + 4i \sqrt{a} \left( \arcsin \sqrt{a - \exp \left( -\frac{1}{2r} \right)} \sqrt{a - e^r} \right) \]
\[ -4ia \left( \text{hypergeom} \left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right], \left[ \frac{3}{2}, \frac{3}{2} \right], a \right) - \exp \left( -\frac{1}{2r} \right) \text{hypergeom} \left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right], \left[ \frac{3}{2}, \frac{3}{2} \right], ae^r \right) - r \exp \left( -\frac{1}{2r} \right) \arcsin \sqrt{ae^r} \right) \]
followed by an even worse one
\[ 2 (8 - 4r + r^2) \sqrt{a - e^r} - 16i \frac{a^{k_2}}{\sqrt{\pi}} \sum_{k_2=0}^{\infty} \frac{a^{k_2}}{2k_2 + 1} \left( 1 - \exp \left( -\frac{1}{2r} (2k_2 + 1) \right) \right) \frac{\Gamma (k_2 + \frac{1}{2})}{\Gamma (k_2 + 1)} \]
\[ +16i \frac{a^{k_2}}{\sqrt{\pi}} \sum_{k_2=0}^{\infty} \frac{a^{k_2}}{(2k_2 + 1)^2} \left( 1 + \frac{1}{2} (-2 - r (2k_2 + 1)) \exp \left( -\frac{1}{2r} (2k_2 + 1) \right) \right) \frac{\Gamma (k_2 + \frac{1}{2})}{\Gamma (k_2 + 1)} \]
\[ -8i \frac{a^{k_2}}{\sqrt{\pi}} \sum_{k_2=0}^{\infty} \frac{a^{k_2}}{(2k_2 + 1)^3} \left( 2 + \frac{1}{3} (-3r (2k_2 + 1) - 6 - \frac{3}{4} r^2 (2k_2 + 1)^2) \exp \left( -\frac{1}{2r} (2k_2 + 1) \right) \right) \frac{\Gamma (k_2 + \frac{1}{2})}{\Gamma (k_2 + 1)} \]
\[ = \text{more hypergeoms!} \]

etc. But still, these integrals can all be generated in a relatively straightforward way, in terms of a particular Gauss hypergeometric function, a.k.a. an incomplete beta function.
\[ \sum_{n=0}^{\infty} \frac{c^n}{n!} \int_0^r \sqrt{a - e^r} n^r \, dr = \int_0^r \sqrt{a - e^r} e^r \, dr = \int_1^{e^r} \sqrt{a - z e^r} \, dz \equiv F (a, c; r) \]
where
\[ \int \sqrt{a - z e^r} \, dz = \frac{\sqrt{a}}{c} e^r \text{hypergeom} \left( \left[ c, -\frac{1}{2} \right], \left[ 1 + c, \frac{z}{a} \right] \right) = \sqrt{a} B_{a/c} (c, 3/2) \]
and where \( z = e^r \). Thus, with due respect for the cut that lurks in the square root, we have
\[ \int_0^r \sqrt{a - e^r} n^r \, dr = \left. \frac{d^n}{dc^n} F (a, c; r) \right|_{c=0} \]
and we thereby obtain another form for the integral in question.
\[ \int_0^r \sqrt{a + br - e^r} \, dr = \sum_{n=0}^{\infty} \frac{b^n}{n!} \left. \frac{\partial^{2n}}{\partial a^n \partial c^n} F (a, c; r) \right|_{c=0} = e^b \frac{d^2}{dc^2} F (a, c; r) \left|_{c=0} \right. \]
The latter “heat kernel” acting on \( F (a, c; r) \) can be cast into an integral form. So, is this any progress, or have we merely gone from one integral representation to another, perhaps more obscure than the first?