PHY612 Problems and Exercises, Spring 2009

Problem 1 Consider a forced harmonic oscillator described by

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 - f(t) \ x$$

where the driving term switches on at $t = 0$. That is to say, $f(t) = 0$ for all $t \leq 0$, while for $t > 0$ the driving force is a reasonably well-behaved but otherwise arbitrary function. Suppose that the system is in the usual oscillator ground state $|n = 0\rangle$ for $t \leq 0$. Compute $\langle n|0, t\rangle$, and $p(n, t) = |\langle n|0, t\rangle|^2$, the amplitude and probability that the force causes the oscillator to make a transition into the $n$th energy eigenstate of the $(f = 0)$ oscillator. Check that your answer gives $\sum_{n=0}^{\infty} p(n, t) = 1$ for arbitrary $f(t)$. For constant $f(t > 0)$, what is $p(n, t)$, and what is the traditional name for this probability distribution? Also compute $\langle n \rangle = \sum_{n=0}^{\infty} n \ p(n, t)$, and compare your answer to the average energy gained by the forced oscillator during the time $t$.

Exercise 1 Restore the mass and coupling parameters to the generator for the classical canonical transformation: $H = p^2 + x \rightarrow -P$. That is to say, find a generator $F(x, X)$, with $p = \partial F(x, X)/\partial x$ and $P = -\partial F(x, X)/\partial X$, that produces the transformation $H = \frac{1}{2m} p^2 - f x = -\gamma P$, for constant $m, f$, and $\gamma$.

Exercise 2 For $t > 0$ the propagator for the Hamiltonian $H = \frac{1}{2m} p^2 - f x$, in position representation with $p = -i \hbar \frac{d}{dx}$, is

$$K(x, X; t) = \sqrt{\frac{m}{2\pi i \hbar t}} \exp \left( i \frac{m}{2\hbar} \frac{(x - X)^2}{t} + i \frac{f}{2\hbar} (x + X) t - i \frac{1}{24\hbar m f^2 t^3} \right)$$

Here, the parameter $f$ is a constant. Verify that this solves the time-dependent Schrödinger equation in the position representation.

Exercise 3 Find the classical action for a trajectory connecting $x_1$ at time $t_1$ to $x_2$ at time $t_2$ for a particle whose dynamics are given by $L = \frac{m}{2} \left( \frac{dx}{dt} \right)^2 + f(t) \ x$. Note that the driving term, i.e. $f(t)$, may have an explicit time dependence. You may assume that $f(t)$ is differentiable, and otherwise reasonably well behaved.

Exercise 4 Consider the Heisenberg picture Hamiltonian $H(t) = \frac{1}{2m} \ p(t)^2 - f(t) \ x(t)$, where $x(t)$ and $p(t)$ are canonically conjugate operators at time $t$. Show that $H(t) = H(0) - \int_0^t x(\tau) \frac{df}{d\tau} d\tau$

Exercise 5 Show that

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n = \frac{t^n}{n!}$$

More generally, for the case $[F(t_1), F(t_2)] = 0$, for all $0 \leq t_1, t_2 \leq t$, show that

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n F(\tau_1) F(\tau_2) \cdots F(\tau_n) = \frac{1}{n!} \left( \int_0^t d\tau \ F(\tau) \right)^n$$
Exercise 6 Specifically, for any well-behaved numerically valued function, $f(\tau)$, establish the integral identities

$$
\left( \int_{t_1}^{t_2} f(\tau) \, d\tau \right)^2 = 2 \int_{t_1}^{t_2} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2 \, f(\tau_2)
$$

$$
\left( \int_{t_1}^{t_2} (\tau - t_1) f(\tau) \, d\tau \right)^2 = 2 \int_{t_1}^{t_2} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2 \, (\tau_2 - t_1) f(\tau_2)
$$

Should it be unclear how to proceed, you might begin by comparing the LHS and RHS for the special case $f(\tau) = a \tau^n + b \tau^m$.

Exercise 7 Show that the fluctuation prefactor $\mathcal{F}$ occurring in the propagator for a linearly driven particle, as discussed in class, satisfies the first order equation

$$
i\hbar \frac{d}{dt_2} \mathcal{F}(t_2, t_1) = \frac{1}{2m} \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (\tau - t_1) f(\tau) \, d\tau \right)^2 \mathcal{F}(t_2, t_1)
$$

Exercise 8 For the Schrödinger picture Hamiltonian $H_S = \frac{1}{2m} p^2 - f(t) x$, use the results of the previous two exercises, and provide all the details, to show complete cancelation of the $x$ and $p$ independent coefficients that arise in the time-dependent Schrödinger equation obeyed by the operator-valued evolution transformation, $U(t_2, t_1)$, as given in class.

Exercise 9 Find the classical action for a SHO particle trajectory connecting $x_1$ at time $t_1$ to $x_2$ at time $t_2$ where the dynamics are given by $L = \frac{m}{2} \left( \frac{dx}{dt} \right)^2 - \frac{1}{2} m\omega^2 x^2$, and where $m$ and $\omega$ are constants.

Exercise 10 Fill in all the algebraic steps to establish the $x$ and $p$ operator identity discussed in lecture:

$$
\exp \left( -\frac{i}{\hbar} t \left( \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 x^2 \right) \right) = \exp \left( -\frac{im\omega}{2\hbar} \tan \frac{\omega t}{2} \, x^2 \right) \exp \left( -\frac{i}{2m\hbar \omega} \sin (\omega t) \, p^2 \right) \exp \left( -\frac{im\omega}{2\hbar} \tan \frac{\omega t}{2} \, x^2 \right)
$$

Exercise 11 Show that the propagator for the SHO follows by taking the $\langle x_1 | \cdots | x_2 \rangle$ matrix element of this last operator identity, and explicitly check that your result for the propagator obeys the time-dependent SHO Schrödinger equation in the position representation.

Exercise 12 Generalize the operator identity for the SHO, as given above, to the case of an oscillator moving in $D$ dimensions. Allow both $m$ and $\omega$ to be “anisotropic.” What is the corresponding expression for the propagator?

Exercise 13 Explicitly evaluate the “time-sliced path integral” for the free particle to obtain, in the limit of an infinite number of equally spaced slices, the result for the free particle propagator.

$$
K(x, X; t) = \sqrt{\frac{m}{2\pi i\hbar t}} \exp \left( i \frac{m}{2\hbar} \frac{(x - X)^2}{t} \right)
$$

Generalize this result to describe a free particle moving in $D$ dimensions.
Exercise 14 Perform the intermediate \( p_k \) \((k = 1, \ldots, N + 1)\) momentum integrations in the time-sliced phase-space form of the functional integral for the propagator, for the intermediate Hamiltonians

\[
H(x_k, p_k; t_k) = \frac{1}{2m} p_k^2 + A(x_k; t_k) p_k + V(x_k; t_k)
\]

to obtain the intermediate Lagrangians \( L(x_k, x_{k-1}; t_k) \) that appear in the remaining time-sliced path integral.

Exercise 15 Continue the integral relation

\[
\frac{1}{\pi^{s/2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \int_0^\infty t^{\frac{s}{2}-1} \Theta(t) \, dt,
\]

valid for \( \Re s > 1 \), to show the relation

\[
\frac{1}{\pi^{s/2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left( t^{\frac{s}{2}-1} + t^{-\frac{s}{2}-\frac{1}{2}} \right) \Theta(t) \, dt,
\]

valid for all \( s \in \mathbb{C} \). Here we have used a “theta function” defined by the sum \( \Theta(t) = \sum_{n=1}^\infty e^{-n^2 \pi t} \). (Hint: Use the “Jacobi transformation” identity \( 1 + 2\Theta(t) = \frac{1}{\sqrt{t}} (1 + 2\Theta(\frac{1}{t})) \).)

Exercise 16 The 3-dimensional SHO radial propagator for fixed angular momentum \( \ell \) is given by

\[
K_\ell(r', t'; r, t) = \frac{m \omega \sqrt{r r'}}{i h \sin(\omega (t' - t))} I_{\ell + \frac{1}{2}} \left( \frac{m \omega r r'}{i h \sin(\omega (t' - t))} \right) \exp \left( \frac{im \omega}{2h} (r^2 + r'^2) \cos(\omega (t' - t)) \right) \sin(\omega (t' - t)).
\]

Show that this obeys the time-dependent, radial Schrödinger equation with potential \( V_{\text{effective}}(r) = \frac{\hbar^2 \ell(\ell+1)}{2mr^2} + \frac{1}{2} m \omega^2 r^2 \).

Exercise 17 Expand the modified Bessel function in the previous result as a sum of Laguerre polynomial bilinears, hence obtain the radial wave functions for the oscillator.

Exercise 18 Let \( r = R^2 \) to show \( \frac{1}{r} \frac{d}{dr} (rf(r)) = \frac{1}{4m\hbar^2} \frac{d^2}{dr^2} \left( R^{3/2} f(R^2) \right) - \frac{3}{16m} f(R^2) \).

Exercise 19 Change variables from \( r, t \) to \( R = \sqrt{r}, s = t R \) in the time-dependent, radial Schrödinger equation for the Coulomb potential. What is the resulting partial differential equation?

Exercise 20 Use the projection of the 3-sphere onto Euclidean 3-space, for which \( d\Omega = \left( \frac{2}{1 + r^2} \right)^3 d^3r \), to compute the total solid angle on the 3-sphere: \( \Omega = \int d\Omega \).

Exercise 21 For the harmonic oscillator described by \( H(x, p) = \frac{1}{2} (x^2 + p^2) \), solve the eigenvalue equation \( H \ast f_n = (n + \frac{1}{2}) \hbar f_n = f_n \ast H \) to obtain

\[
f_n(x, p) = \frac{(-1)^n}{\pi \hbar} L_n \left( \frac{2}{\hbar} (x^2 + p^2) \right) e^{-(x^2 + p^2)/\hbar}
\]

where the Laguerre polynomials are given by \( L_n(z) = \frac{1}{n!} e^{z} \frac{d^n}{dz^n} (z^n e^{-z}) \).
Exercise 22 For a pure state Wigner function normalized such that \( \int dx dp \, f(x, p) = 1 \), prove that
\[
-\frac{1}{\pi \hbar} \leq f(x, p) \leq \frac{1}{\pi \hbar}.
\]

Exercise 23 Use the integral form of the \( \ast \) product to show
\[
\exp \left( -\frac{a}{\hbar} \left( x^2 + p^2 \right) \right) \ast \exp \left( -\frac{b}{\hbar} \left( x^2 + p^2 \right) \right) = \frac{1}{1 + ab} \exp \left( -\frac{a + b}{(1 + ab) \hbar} \left( x^2 + p^2 \right) \right)
\]
Hence two Gaussians \( \ast \)-commute with one another if they are centered on the same point (here, the origin of phase space). Note that these are ordinary exponentials, not \( \ast \)-exponentials. Were the latter involved, we would have just \( \exp \left( -\frac{a}{\hbar} \left( x^2 + p^2 \right) \right) \ast \exp \left( -\frac{b}{\hbar} \left( x^2 + p^2 \right) \right) = \exp \left( -\frac{a + b}{\hbar} \left( x^2 + p^2 \right) \right) \).

Exercise 24 For the SHO with \( H(x, p) = \frac{1}{2} \left( x^2 + p^2 \right) \),
\[
\exp \left( itH(x, p) / \hbar \right) = \frac{1}{\cos (t/2)} \exp \left( \frac{i}{\hbar} \left( x^2 + p^2 \right) \tan (t/2) \right).
\]
Verify this by showing that the explicit function on the RHS is a solution of
\[
H(x, p) \ast \exp \left( itH(x, p) / \hbar \right) = -i\hbar \frac{\partial}{\partial \tau} \exp \left( itH(x, p) / \hbar \right) = \exp \left( itH(x, p) / \hbar \right) \ast H(x, p)
\]
with initial condition \( \exp \left( itH(x, p) / \hbar \right)|_{t=0} = 1 \).

Exercise 25 Another way to present time evolution in phase space is to compute the “Wigner transform” of the propagator in coordinate space (\( \hbar \equiv 1 \) here)
\[
K(x, p; X, P; t) = \frac{1}{2\pi} \int dy dY \, K \left( x + \frac{1}{2} y, X + \frac{1}{2} Y; t \right) e^{-ipy + ipY} K^\ast \left( x - \frac{1}{2} y, X - \frac{1}{2} Y; t \right),
\]
so that the Wigner function evolves as
\[
f(x, p; t) = \int dX dP \, K(x, p; X, P; t) \, f(X, P; 0).
\]
For the SHO of the previous problem, where
\[
K(x, X; t) = \frac{1}{\sqrt{2\pi i \sin t}} \exp \left( \frac{i}{2} \left( x^2 + X^2 \right) \cot t \right) \frac{ixX}{\sin t},
\]
evaluate the Wigner transform to show that the phase space propagator is simply Dirac deltas for the classical motion.
\[
K(x, p; X, P; t) = \delta (x \cos t - p \sin t - X) \, \delta (x \sin t + p \cos t - P).
\]

Exercise 26 For the linear potential problem, where \( H = p^2 + x \), obtain in closed form the Wigner energy eigenfunctions \( f_E(x, p) \), such that \( H \ast f_E(x, p) = E f_E(x, p) \). \( H \ast f_E(x, p) = f_E(x, p) \ast H \). Note the energy spectrum is continuous, \( -\infty < \sigma (H) < +\infty \), so the Wigner functions will have a continuum normalization. From the explicit form for \( f_E(x, p) \), or otherwise, show that \( f_{E_1}(x, p) \ast f_{E_2}(x, p) = 0 \) if \( E_1 \neq E_2 \).
Exercise 27  What is the phase-space propagator \( \mathcal{K}(x,p;X,P;t) \) for the linear potential problem?

Exercise 28  Express the star exponential \( \exp_\star (\alpha (p^2 + x)) \) in terms of the ordinary exponential \( \exp (\beta (p^2 + x)) \). What is the relation between \( \alpha \) and \( \beta \)?

Exercise 29  Solve \( i\partial_t U(t) = H \ast U(t) = U(t) \ast H \) for the linear potential problem, with \( U(0) = 1 \), in terms of ordinary functions.

Exercise 30  Show that
\[
\frac{1}{1 + \tau \partial_\sigma} \cos (a \sigma) = \frac{1}{1 + (a\tau)^2} \left( \cos (a \sigma) + a \tau \sin (a \sigma) \right), \\
\frac{1}{1 + \tau \partial_\sigma} \sin (a \sigma) = \frac{1}{1 + (a\tau)^2} \left( \sin (a \sigma) - a \tau \cos (a \sigma) \right).
\]

Exercise 31  Work out the “twisted” field phase-space propagator \( \mathcal{K}[x(\sigma), p(\sigma); X(\sigma), P(\sigma); \tau] \) for the linear potential problem, where the field interacts with an external source that may vary with position, i.e. the interaction energy is \( V = \int f(\sigma) x(\sigma) \, d\sigma \). Compare your result to the conventional field propagator \( \mathcal{K}[x(\sigma), p(\sigma); X(\sigma), P(\sigma); t] \) for the same problem. What is the relation between \( \tau \) and \( t \)?

Exercise 32  For the \( su(2) \) algebra of the classical \( (m = 1 = \omega) \) oscillator, as realized by
\[
J_x = \frac{1}{2} (p_x p_y + xy), \quad J_y = \frac{1}{2} (p_y^2 + y^2 - p_x^2 - x^2), \quad J_z = \frac{1}{2} (xp_y - yp_x),
\]
show that
\[
J_x^2 + J_y^2 + J_z^2 = \frac{1}{16} (p_y^2 + p_x^2 + x^2 + y^2)^2.
\]

Hence establish that Poisson bracket time-evolution on the phase space is equivalent to Nambu 4-bracket evolution, in the following sense.
\[
\{f, J_x^2 + J_y^2 + J_z^2\}_{PB} = 2 \times \{f, J_x, J_y, J_z\}_{4NB}.
\]

The classical 4-bracket here is defined by the Jacobian
\[
\{A, B, C, D\}_{4NB} = \left( \frac{\partial (A, B, C, D)}{\partial (x, p_x, y, p_y)} \right) = \text{det} \begin{pmatrix}
\frac{\partial A}{\partial x} & \frac{\partial A}{\partial p_x} & \frac{\partial A}{\partial p_y} & \frac{\partial A}{\partial y} \\
\frac{\partial B}{\partial x} & \frac{\partial B}{\partial p_x} & \frac{\partial B}{\partial p_y} & \frac{\partial B}{\partial y} \\
\frac{\partial C}{\partial x} & \frac{\partial C}{\partial p_x} & \frac{\partial C}{\partial p_y} & \frac{\partial C}{\partial y} \\
\frac{\partial D}{\partial x} & \frac{\partial D}{\partial p_x} & \frac{\partial D}{\partial p_y} & \frac{\partial D}{\partial y}
\end{pmatrix}.
\]