Free particle Consider a free classical particle of unit mass $m = 1$ moving on the $xy$-plane. This is of course an integrable system. The motion is described by explicit functions of time

$$x(t) = x_0 + p_x t, \quad y(t) = y_0 + p_y t,$$

with $p_x$ and $p_y$ constant. Alternatively, the system is described by a one-dimensional curve in the four-dimensional $(x, y, p_x, p_y)$ phase-space, the latter a direct product of position and momentum planes. For example

\[ \text{A free particle path on the position plane.} \quad \text{A free particle point on the momentum plane.} \]

The direct product of the straight line in position space and the point in momentum space is a straight line in the phase-space. This emphasizes an essentially geometrical view of the phase-space, and integrability\(^1\).

\(^1\)An analytic approach to the question of integrability, using complex analysis, was first given by Sofia Kovalevskaya in 1890, in her solution of the motion of an $I_x = I_y = 2I_z$ gyroscope in a gravitational field. In many ways her work was the forerunner of modern investigations of integrability, but is too complex (pun not intended) to discuss here. Unfortunately, her life (1850-1891) and career were cut short by illness (see [http://www.agnescott.edu/triddle/women/kova.htm](http://www.agnescott.edu/triddle/women/kova.htm)).

We will use very rudimentary complex variable methods in our discussion of exponential potentials below.
Moreover, the free particle problem is maximally super-integrable\textsuperscript{2}.

There are two obvious constants of the motion here, $p_x$ and $p_y$, which are in involution, $\{p_x, p_y\} = 0$. But, actually, there are more than two independent constants of the motion for this problem: e.g. a third invariant is $L = xp_y - yp_x$, although it is not in involution with the other two, $\{L, p_x\} = p_y$ and $\{L, p_y\} = -p_x$.

That is to say, three explicitly solvable time-independent phase-space constraints can be found which reduce the four-dimensional phase-space down to a one-dimensional curve. The third invariant may in fact be used to obtain

$$y(x) = \frac{1}{p_x} (xp_y - L)$$

where all the quantities on the RHS are constant except $x$. Hence the phase-space plots above. This is exactly the same result as follows from solving the first of the above time-parameterized equations for

$$t(x) = \frac{x - x_0}{p_x}$$

and then substituting into the second equation to obtain $y(x) = y(t(x)) = y_0 + p_y \left( \frac{x - x_0}{p_x} \right) = \frac{1}{p_x} (xp_y - L)$ since $L = x_0 p_y - y_0 p_x$.

In summary for the free particle on the plane, our choice for the three phase-space invariants are

$$p_x = C_1, \quad p_y = C_2, \quad xp_y - yp_x = C_3.$$\noindent In general, for $N$ degrees of freedom, if we are able to explicitly obtain $N$ invariants ("actions") in involution on the phase-space, the model is said to be integrable in the Liouville sense. If we can obtain additional invariants up to $2N - 1$ in total, the model is said to be super-integrable. For the particle on the plane, $N = 2$ corresponding to $x$ and $y$, so we have at most $2N - 1 = 3$ invariant relations. The free particle illustrates the maximally super-integrable situation.

A model is maximally super-integrable if $t$ is supplanted, say, by one “angle” variable in the explicit expressions for the remaining $N - 1$ angle variables, but it really is not necessary to first find and then eliminate the $t$ dependence to find the additional invariants.

Let’s consider another classic example to further illustrate these points.

\textsuperscript{2}The ideas here go back at least to A. Sommerfeld (pictured, left), but became an active area of research following the work of P. Winternitz (pictured, right), et al., in 1965. The term “superintegrable” is due to S. Wojciechowski in 1983.
**Kepler problem in phase-space** The bound orbits are fixed ellipses (Kepler’s 1st law). Therefore, for any particular orbit we have *two* phase-space constraints\(^3\). In terms of conveniently chosen coordinates (Runge-Lenz-Pauli-vector fanatics who prefer coordinate-free statements should see R. Chatterjee, hep-th/9501141):

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x}{a^2} p_x + \frac{y}{b^2} p_y = 0
\]

So we have reduced phase-space to a two dimensional surface: locally, elliptical \((x, y)\) curve \(\otimes\) straight \((p_x, p_y)\) line.

\(^3\)Note that these are *not* in involution under Poisson brackets. This poses no real difficulties.
We require one more constraint to reduce the phase-space to a one-dimensional curve. There are obviously many ways (an $\infty$ choice!) to do this, but two ways stand out historically. Both are additional linear constraints on the momenta. These are the two classic central force cases investigated by Isaac Newton, in *The Principia*, Book I, Section II, Propositions X and XI: namely, the SHO and the Kepler problem\(^4\).

The last diagram is Kepler’s ellipse drawn for Mars, from *Astronomia Nova* (1609).

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\(^4\)There is one other famous case of elliptical motion besides the SHO and the Kepler problem, namely Jacobi’s free particle on an ellipsoid (specialized to just an ellipse). For Jacobi’s example, an additional quadratic invariant is just the free particle kinetic energy $p_x^2 + p_y^2$. The force on the particle is not central in this case, but is always normal to the ellipse, and hence angular momentum is not conserved. Instead, a function of momentum and $L$ about the origin is conserved: e.g. $A = p_x^2 + (xp_y - yp_x)^2 / (a^2 - b^2)$.
The simple harmonic oscillator results from an invariant \( xp_y - yp_x = L \) where \( L \) is the *angular momentum* computed about the center of the ellipse. The inverse-square force problem results from (Kepler’s 2nd law)

\[
(x \pm c) p_y - yp_x = L, \quad \text{where} \quad c = \sqrt{a^2 - b^2}
\]

Taking \( \pm c \) corresponds to computing the angular momentum about the left/right focus of the ellipse.

![Diagram of ellipse with phase-space coordinates]

The phase-space of the particle is now reduced to a one-dimensional curve. Usually this curve in phase-space is parameterized by the time \( t \). But once again we have three solvable constraints, so the system is again maximally super-integrable, and we can immediately solve for all phase-space variables in terms of a single dynamical variable, say \( x \).

\[
y(x) = \pm b \sqrt{1 - x^2/a^2}, \quad p_x(x) = \frac{-y(x) L/b^2}{1 + xc/a^2}, \quad p_y(x) = \frac{xL/a^2}{1 + xc/a^2}.
\]
Here we have solved the left focus case \( L = (x + c)p_y - yp_x \) of the linear equations for the momenta, using matrices

\[
\begin{pmatrix}
\frac{x}{a^2} & \frac{y}{b^2} \\
-y & x + c
\end{pmatrix}
\begin{pmatrix}
p_x \\
py
\end{pmatrix}
= \begin{pmatrix}
0 \\
L
\end{pmatrix}
\]

Matrix equations are so organized, they clearly suggest the involvement of either accountants, or lawyers\(^5\).

But I digress ...

\(^5\)These two pictures are not of the same person, say before and after the Taliban left town. Rather they are (left to right) James Sylvester and Arthur Cayley, who became acquainted as lawyers in London, in the 1850’s, and in their free time created the theory of matrices. This development might not have happened but for some crucial stimulation by mathematics students in pre-Civil War Virginia.
\[
\begin{align*}
\det \begin{pmatrix} x/a^2 & y/b^2 \\ -y & x + c \end{pmatrix} \bigg|_{x^2/a^2 + y^2/b^2 = 1} &= 1 + xc/a^2 = \frac{R(x)}{a} \\
\end{align*}
\]

This is a particularly nice fact given undergraduates’ abhorrence of spherical coordinates\(^6\).

\(^6\) Another way to say this, perhaps more familiar from textbooks, is

\[ R = a \frac{(1 - e^2)}{1 - e \cos \theta} = \frac{b^2}{a - e \cos \theta}, \]

where the eccentricity of the orbit is \(e = c/a\), and where \(\theta\) measures the angle between the vector \(R\) and the semi-major axis (our choice of angle has \(\theta = \pi\) when the orbiting body is at closest approach, or “periapsis”). The relation to the \(\det\) expression follows from \(x + c = R \cos \theta\). Hence \(aR - c(x + c) = b^2\), or \(aR = a^2 + cx\).
Time evolution, as Hamilton’s flow  This is the usual approach to time evolution.

\[ H(x,p) \] defines a vector field on the phase-space, tangent to all trajectories, through the Poisson bracket.

\[ \frac{df}{dt} = \{f, H\} \]

where, for example, in one dimension, \( \{f, H\} = \frac{\partial f}{\partial x} \frac{\partial H}{\partial p} \), etc. Everyone understands this fairly well.

What is not so widely appreciated, perhaps, is that the above description of the Kepler orbits as constraints in phase-space immediately leads to the \( 1/R \) potential, hence the inverse-square force, and the usual Hamiltonian after nothing but a bit of algebra. No actual integrations are required, much to the relief of freshman physics students! That is to say, the inverse-square force follows immediately from the assumption that there is a potential energy, and the first two laws of Kepler: elliptical orbits (1st law) and conserved angular momentum about the attracting focus (2nd “equal areas” law). In this sense, Kepler’s third law \( (T^2 \propto a^3) \) is a consequence of the first two.

To see this just compute the kinetic energy from the previous three phase-space constraints, starting with

\[ p_x = \frac{-yL/b^2}{1 + xc/a^2} , \quad p_y = \frac{xL/a^2}{1 + xc/a^2} . \]

\[ K = \frac{1}{2m} (p_x^2 + p_y^2) = \frac{L^2}{2m(1 + cx/a^2)^2} \left( x^2/a^4 + y^2/b^4 \right) . \]

Then replace \( y^2 = b^2 \left( 1 - x^2/a^2 \right) \), and express everything as a function of \( x \). After some elementary algebra, the result is just

\[ K = \frac{L^2}{2mb^2} \left( \frac{2a}{R(x)} - 1 \right) . \]
From this it is evident that only a $1/R$ potential $U$ will conserve the total energy $E$ in the Kepler case. $E = K + U$, or alternatively,

$$U = E - K.$$ 

Since $E$ is a constant, $L$ is a constant, and $a$, $b$, and $c$ are constants for a given elliptical orbit, the only position dependence is in the $1/R$ term in $K$. That is,

$$U = E + \frac{L^2}{2mb^2} \left(1 - \frac{2a}{R}\right).$$

Therefore, we have deduced from the first and second laws of Kepler, and conservation of energy, that the potential energy goes like the inverse of the distance, up to an overall irrelevant additive constant. Accordingly, the force will go like the inverse square of the distance.

You may also look at it this way. The Kepler orbit example is the first case of an inverse problem: given the orbits, find the potential. The solution, as above, is purely algebraic.

If we choose for $U$ to vanish as $R \to \infty$, which is customary, then we have a relation between $E$, $L$, and $b$. Namely

$$E = \frac{L^2}{2mb^2}.$$ 

Further identifying the parameters in the potential with those of Newton, $U = -GMm/R$, we have the additional relation

$$GMm = \frac{aL^2}{mb^2}.$$ 

Thus, another way to express the total energy of the mass $m$ is

$$E = -\frac{GMm}{2a}.$$ 

The total energy is negative here, as you may recall, because the mass $m$ is in a bounded orbit and cannot escape to infinity.