4 Third reading assignment, partial differential equations

Read M&W, Chapters 4 and 8, or their equivalent in other texts. It might be good to first review Nearing, Chapters 5, 10, and 15.

Now, the theta function in (39) satisfies an obvious, but very interesting, partial differential equation: The one (spatial) dimension “diffusion” or “heat” equation.

\[
\frac{\partial}{\partial t}\vartheta(x,t) = \frac{\partial^2}{\partial x^2}\vartheta(x,t)
\]

This follows, at least for \( t > 0 \), since each term in the series representation \( \vartheta(x,t) = \sum_{n=-\infty}^{\infty} e^{in\sqrt{\pi}x} e^{-n^2\pi^2 t} \) satisfies this equation.

\[
\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) e^{in\sqrt{\pi}x-n^2\pi^2 t} = 0
\]

In physical problems, the heat equation appears with a relevant scale factor \( \lambda \), which we take to be real and positive, for now. For example, in the case of temperature and heat flow problems in one spatial dimension \( x \), the space and time dependent temperature field satisfies

\[
\frac{\partial}{\partial t} T(x,t) = \lambda \frac{\partial^2}{\partial x^2} T(x,t)
\]  

(43)

where \( \lambda = K/C\rho \) with \( K \) being the thermal conductivity, \( C \) the specific heat, and \( \rho \) the density of the underlying material, all assumed to be constants. Eqn (43) is an example of a parabolic partial differential equation in the two variables \( x \) and \( t \).

A typical boundary value problem involving (43) is to give \( T(x, t = 0) \) for all \( x \in \mathbb{R} \), and then determine \( T(x, t \neq 0) \). Formally, the answer is given immediately by the time power series

\[
T(x, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial^n}{\partial x^n} T(x, 0) \bigg|_{t=0}
\]

If this is to be a solution to (43), every time derivative reduces to two spatial derivatives, hence

\[
T(x, t) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \frac{\partial^{2n}}{\partial x^{2n}} T(x, t = 0)
\]  

(44)

and all of the spatial derivatives appearing in the series are determined by the initial data. For now, we assume the initial function is infinitely differentiable with respect to \( x \), i.e. that \( T(x, 0) \) is a “good function” in the sense described in J.M. Lighthill, Fourier Analysis and Generalized Functions. Thus the time series solution to (43) is completely determined, if it exists, i.e. if the series converges.

A short-hand way to write (44) is obviously

\[
T(x, t) = e^{\lambda t \frac{\partial^2}{\partial x^2}} T(x, 0)
\]  

(45)

Through the use of Fourier analysis and the theory of distributions, this formal relation actually becomes a useful practical means of determining \( T(x, t) \). Write

\[
T(x, 0) = \int_{-\infty}^{+\infty} T(X, 0) \delta(x-X) dX
\]

where \( \delta \) is the Dirac delta, as usefully represented here by

\[
\delta(x-X) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-X)} dk
\]

(44)

Then, again within the framework of the theory of distributions, we have

\[
e^{\lambda t \frac{\partial^2}{\partial x^2}} T(x, 0) = \int_{-\infty}^{+\infty} T(X, 0) e^{\lambda t \frac{\partial^2}{\partial x^2}} \delta(x-X) dX
\]

55
But (see HW#8)

\[ e^{\lambda t} \frac{\partial}{\partial x} \delta(x - X) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\lambda tk^2} e^{ik(x-X)} dk = \frac{e^{-(x-X)^2/(4\lambda t)}}{\sqrt{4\pi \lambda t}} \equiv g_0(x, X; t) \]

This is known as the “heat kernel” or “diffusion Green function” for the problem under discussion. Note that in this case the kernel depends only of the difference of the two spatial positions, \( g_0(x, X; t) = g_0(x - X; t) \).

This result for \( g_0(x, X; t) \) gives the solution an explicit integral form for \( t \neq 0 \). Eqn(45) becomes

\[ T(x, t) = \frac{1}{\sqrt{4\pi \lambda t}} \int_{-\infty}^{+\infty} T(X, 0) e^{-(x-X)^2/(4\lambda t)} dX \]  

(46)

You may consider this as a Gaussian transform of the initial data.

As an example that reveals both the power of the method, as well as the pitfalls of parabolic PDEs, suppose the initial data is itself given by a Gaussian.

\[ T(x, 0) = T_0 e^{-x^2/L^2} \]

Inspection of the integrand in (46) shows that the integral exists only for \( \frac{1}{4\lambda t} + \frac{1}{L^2} > 0 \), i.e. \( 4\lambda t > -L^2 \).

Evaluation of the integral (again see HW#8) gives an analytic result

\[ T(x, t) = \frac{T_0}{\sqrt{1 + 4\lambda t/L^2}} e^{-\frac{x^2}{4\lambda t + \frac{1}{L^2}}} \]  

(47)

and indeed, there is a essential singularity from the exponential at \( 4\lambda t = -L^2 \). This singularity prevents the initial data from being evolved to earlier times. Forward evolution to all \( t > 0 \) is not a problem, however, as evident in the final answer. The initial data broadens or spreads and has its maximum value at \( x = 0 \) diminished as time progresses, \( T(x = 0, t) = \frac{T_0}{\sqrt{1 + 4\lambda t/L^2}} \). That is to say, the heat “diffuses” through the material and the initially hottest region cools.

The series expansion of (47) in powers of \( t \) for fixed \( x \) gives explicitly (44) for this example. The series converges for \( |t| < L^2/(4\lambda t) \). The first few terms are

\[ \frac{T_0}{\sqrt{1 + 4\lambda t/L^2}} e^{-\frac{x^2}{4\lambda t + \frac{1}{L^2}}} = T_0 e^{-\frac{x^2}{4\lambda t + \frac{1}{L^2}}} \left( 1 + \frac{2x^2 - L^2}{4\lambda t} + \frac{12x^2 - 4x^4 + 3L^4}{16\lambda t} + \frac{490x^2L^4 - 60x^4L^2 + 8x^6 - 15L^6}{12^2} + O(t^4) \right) \]

but this is not particularly informative. It would perhaps be more useful to expand the solution in powers of \( x \) for fixed \( t \), thereby obtaining a time Laurent expansion in powers of \( \frac{1}{\sqrt{4\lambda t + \frac{1}{L^2}}} \).

**HW Problem 8:** Show the following.

\[ \int_{-\infty}^{+\infty} e^{ikx} \frac{1}{\cosh x} dx = \frac{\pi}{\cosh \frac{k}{2\pi}} \quad , \quad \int_{-\infty}^{+\infty} e^{ikx} e^{-x^2} dx = e^{-\frac{k^2}{4}} \sqrt{\pi} \]

\[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} e^{-\lambda tk^2} dk = \frac{e^{-x^2/(4\lambda t)}}{\sqrt{4\pi \lambda t}} \]

and

\[ \int_{-\infty}^{+\infty} e^{-(x-X)^2/(4\lambda t)} \frac{1}{\sqrt{4\pi \lambda t}} e^{-X^2/L^2} dX = \frac{L}{\sqrt{L^2 + 4\lambda t}} e^{-\frac{x^2}{4\lambda t + \frac{1}{L^2}}} \]

**Solution:**

To show \( \int_{-\infty}^{+\infty} e^{ikx} \frac{1}{\cosh x} dx = \frac{\pi}{\cosh \frac{k}{2\pi}} \), just modify slightly the discussion in M&W on pp 71 and 72.
To show \( \int_{-\infty}^{+\infty} e^{ikx} e^{-x^2} \, dx = e^{-\frac{1}{4}k^2} \sqrt{\pi} \), complete the square in the exponent of the integrand, and use 
\[ I = \int_{-\infty}^{+\infty} e^{-x^2} \, dx = \sqrt{\pi}, \]
which in turn can be shown by considering \( I^2 = \int_{-\infty}^{+\infty} e^{-x^2} \, dx \int_{-\infty}^{+\infty} e^{-y^2} \, dy \) and converting to polar coordinates on the resulting plane of integration.

Similarly, complete the square in the exponents of the integrands for the last two integrals.

Keeping in mind that the equation under investigation is linear, singularities such as that in (47) for \( t < 0 \) cause numerical instabilities and prevent one from reliably evolving real temperature data backward in \( t \), in practice. Imagine some experimentally determined initial temperature data. If there is a small erroneous bump (approximated by a narrow, but low lying Gaussian) in that data, as opposed to the great bulk of the data which may be well-approximated by a much broader Gaussian, and the data in total is described by a sum of those two Gaussians, the error will dominate and produce the earliest singularity encountered if the data is evolved backward in time, just because the erroneous bump is narrower and has a smaller value of \( L \). However, for forward evolution with \( t > 0 \), the erroneous bump dies away faster than the bulk of the data, and so forward evolution is stable in the presence of the error. Among other things, this means that given a good thermometer and a bucket containing a chunk of ice, I can make temperature measurements\(^4\) and accurately predict how much ice will melt during the course of the next several hours. But given a bucket of only water and a thermometer, no matter how precise that thermometer may be, I cannot tell you how much ice was in the bucket only minutes earlier, nor even if there were any ice at all. In fact, from temperature measurements, I cannot even tell if (part of) the water was previously boiling.

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Discuss ... Diffusion/heat equation as an example of a parabolic equation, Green function, heat kernel, separation of variables, Schrödinger equation, and all that.

Another example: Discuss the “hot hypercube” placed in a cold bath. See Nearing for the one-dimensional slab, and M&W for the 3-dimensional cube. A closed form answer involves Jacobi theta functions.

Discuss more examples using (46). While we wrote (44) assuming infinite differentiability of the initial data, the integral form of the solution (46) actually does not require differentiability of that data, in the usual sense. The initial \( T \) could be a generalized function.

Do some cases: initial \( T \) as step function; initial \( T \) as a delta function. The latter permits us to give a simple physical interpretation of the kernel.

\(^4\)Well, in principle, I guess I could make such measurements. But remember, I’m a theorist, and not an experimenter!
For any initial data given as step functions, the resulting function at later times will be given in terms of error functions. These are defined by integrals of Gaussians over a finite domain.

\[
\text{erf} (z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} \, ds , \quad \frac{d}{dz} \text{erf} (z) = \frac{2}{\sqrt{\pi}} e^{-z^2} \\
\text{erf} (\infty) = 1 , \quad \text{erf} (0) = 0 , \quad \text{erf} (-\infty) = -1
\]

Note that erf is an odd function.

\[
\text{erf} (z) = -\text{erf} (-z)
\]

This graph suggests a clear physical interpretation of erf as a smoothed-out step function whose sharp corners have been rounded by a diffusion process.

The complementary error function is defined as the remaining integral out to infinity.

\[
\text{erfc} (z) = 1 - \text{erf} (z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-s^2} \, ds , \quad \frac{d}{dz} \text{erfc} (z) = -\frac{2}{\sqrt{\pi}} e^{-z^2} \\
\text{erfc} (\infty) = 0 , \quad \text{erfc} (0) = 1 , \quad \text{erfc} (-\infty) = 2
\]
Suppose the initial temperature is a constant $T_0$ for the finite interval $a < x < b$ and zero outside this interval. This initial temperature profile is that of a “rectangular pulse.” (We’ll ignore the vertical sides of the pulse, since whatever value we assign to $T(a, t = 0)$ or $T(b, t = 0)$ will not affect the integral to follow.) Then (46) gives the temperature at later times.

\[
T(x, t) = \frac{1}{\sqrt{4\pi\lambda t}} \int_a^b T_0 e^{-\frac{(x-x)^2}{4\lambda t}} \, dx
\]

\[
= \frac{1}{\sqrt{4\pi\lambda}} \left( \int_a^\infty T_0 e^{-\frac{(x-x)^2}{4\lambda t}} \, dx - \int_b^\infty T_0 e^{-\frac{(x-x)^2}{4\lambda t}} \, dx \right)
\]

\[
= \frac{1}{\sqrt{\pi}} \left( \int_{\frac{a-x}{\sqrt{4\lambda t}}}^{\infty} T_0 e^{-s^2} \, ds - \int_{\frac{b-x}{\sqrt{4\lambda t}}}^{\infty} T_0 e^{-s^2} \, ds \right)
\]

\[
= \frac{T_0}{2} \text{erfc} \left( \frac{a-x}{\sqrt{4\lambda t}} \right) - \frac{T_0}{2} \text{erfc} \left( \frac{b-x}{\sqrt{4\lambda t}} \right)
\]

That is to say

\[
T(x, t) = \frac{T_0}{2} \text{erfc} \left( \frac{b-x}{\sqrt{4\lambda t}} \right) - \frac{T_0}{2} \text{erfc} \left( \frac{a-x}{\sqrt{4\lambda t}} \right)
\]

(48)

As a check on this solution, we compare the time and space partial derivatives to see if the heat equation is obeyed. In essence, this boils down to checking

\[
\frac{\partial}{\partial t} \text{erf} \left( \frac{x}{\sqrt{4\lambda t}} \right) = \lambda \frac{\partial^2}{\partial x^2} \text{erf} \left( \frac{x}{\sqrt{4\lambda t}} \right)
\]

This is straightforward to verify using the derivative relation \( \frac{d}{dz} \text{erf}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2} \).

\[
\frac{\partial}{\partial t} \text{erf} \left( \frac{x}{\sqrt{4\lambda t}} \right) = \frac{\partial}{\partial t} \left( \frac{x}{\sqrt{4\lambda t}} \right) \times \frac{2}{\sqrt{\pi}} e^{-x^2/(4\lambda t)} = \frac{-x}{\sqrt{4\lambda \pi t^{3/2}}} e^{-x^2/(4\lambda t)}
\]

\[
\lambda \frac{\partial}{\partial x} \text{erf} \left( \frac{x}{\sqrt{4\lambda t}} \right) = \lambda \left( \frac{1}{\sqrt{4\lambda}} \right) \times \frac{2}{\sqrt{\pi}} e^{-x^2/(4\lambda t)} = \frac{-x}{\sqrt{4\lambda \pi t^{3/2}}} e^{-x^2/(4\lambda t)}
\]

So the heat PDE is satisfied by \( \text{erf} \left( \frac{x}{\sqrt{4\lambda t}} \right) \). The time behavior of \( \text{erf} \left( \frac{x}{\sqrt{4\lambda t}} \right) \) is evident in the following plot.
In this three-dimensional graph, time increases “into the page” showing that the sharp step becomes a more gentle slope as the data evolves. (Numerical values on the time axis are for \( \lambda t \) and not just \( t \).)

Similarly, the rectangular pulse evolves graphically as follows, for various time steps.

Pulse evolution for \( \lambda t = 0.001 \) (red), \( \lambda t = 0.01 \) (orange), \( \lambda t = 0.1 \) (green), \( \lambda t = 0.5 \) (blue)

We took \( T_0 = 1 \) for \(-1 < x < 1\). Note that the total area under the temperature curve is constant in time, corresponding to energy conservation.

**HW Problem #1:** Show that

\[
\frac{d}{dt} \int_{-\infty}^{\infty} T(x, t) \, dx = 0
\]

for the Gaussian and rectangular pulse examples. What criteria are needed for this to be true for other situations?

**Solution:** TO BE WRITTEN ... Also, see HW#12 below.

**HW Problem #2:** As time passes, the rectangular pulse in the above example starts to resemble a simple, smooth bump. (A Gaussian, perhaps?) Explain this.

**Solution:** We consider the large \( t \) asymptotic behavior of the integral solution. Integration by parts is the method of choice.

\[
\int_{a}^{b} e^{-(x-X)^2/(4\lambda t)} \, dX = \int_{a}^{b} d \left( X e^{-(x-X)^2/(4\lambda t)} \right) - \int_{a}^{b} X d \left( e^{-(x-X)^2/(4\lambda t)} \right)
\]

\[
= be^{-(x-b)^2/(4\lambda t)} - ae^{-(x-a)^2/(4\lambda t)} + \frac{1}{2\lambda t} \int_{a}^{b} X (X-x) e^{-(x-X)^2/(4\lambda t)} \, dX
\]

Continuing in this way, we generate an asymptotic series in powers of \( \frac{1}{t} \). Keeping only the first two terms gives

\[
T(x, t) \sim_{t \to \infty} \frac{T_0}{\sqrt{4\pi\lambda t}} \left( b \left( 1 + \frac{b(b-x)}{2\lambda t} \right) e^{-b(b-x)^2/(4\lambda t)} - a \left( 1 + \frac{a(a-x)}{2\lambda t} \right) e^{-(x-a)^2/(4\lambda t)} \right) + O \left( \frac{1}{t^{3/2}} \right)
\]
This is not too complicated, although it is not just a simple Gaussian. For comparison purposes, we plot the leading asymptotic term and the leading + first correction terms along with the exact answer, for the previous specific numerical example, at $\lambda t = 1$ and at $\lambda t = 5$.

![Graph](image1.png)

Exact (solid line) versus leading asymptotic term (dashes) for $\lambda t = 1$ (red), $\lambda t = 5$ (blue).

![Graph](image2.png)

Exact (solid line) versus leading + next asymptotic terms (dashes) for $\lambda t = 1$ (red), $\lambda t = 5$ (blue).

Note that the leading term asymptotic approximation has the virtue of always being positive (as required for a physical temperature) whereas the leading + first correction is actually negative for larger values of $|x|$. Moreover, the leading term alone is a closer numerical match to the exact answer, even at the central peak, for both of the times chosen. Evidently, this is one of those cases where it was not worth the effort to go beyond the leading term!
An interesting special case of (48) is found by taking the interval to be the half-line with, say, \( T(x < 0, t = 0) = T_0, \ T(x > 0, t = 0) = 0 \). Then for \( t > 0 \)

\[
T(x, t) = \frac{T_0}{2} \left\{ 1 - \text{erf} \left( \frac{x}{\sqrt{4\lambda t}} \right) \right\}
\]

This has the interesting feature that \( T(x = 0, t) = \frac{T_0}{2} \) for all \( t > 0 \). Thus, if we simply ignore the region \( x < 0 \), we have serendipitously solved the heat equation with non-trivial Dirichlet conditions given on the L-shaped open boundary \( \{ x \geq 0, t = 0 \} \cup \{ x = 0, t > 0 \} \). For convenience, we call the fixed value on the \( x = 0 \) line \( T_1 = T_0/2 \). We then have

\[
T(x, t) = T_1 \text{erf} \left( \frac{x}{\sqrt{4\lambda t}} \right) \quad \text{solves} \quad \frac{\partial}{\partial t} T(x, t) = \lambda \frac{\partial^2}{\partial x^2} T(x, t)
\]

with boundary conditions \( T(x > 0, t = 0) = 0 \), \( T(x = 0, t > 0) = T_1 \)

Exactly what happens at the corner \( x = 0, t = 0 \) is left for the reader to decide. This problem is also discussed in Mathews and Walker, Chapter 8, pp 240-242, using Laplace transform methods. (Note that our \( T_1 \) is the same as M\&W’s \( T_0/2 \).

5 Fourth reading assignment, more on PDEs and special functions

Read M\&W, Chapter 7, or its equivalent in other texts. Especially, see Temme’s book.

The last example suggests looking at more complicated boundary conditions. These can also be handled through the use of Green functions which incorporate the salient features of the boundary conditions. As illustration, again take the L-shaped boundary \( \{ x = 0, t > 0 \} \cup \{ x \geq 0, t = 0 \} \), and again assume Dirichlet boundary conditions, only this time take \( T(x > 0, t = 0) \) to be some given function, not necessarily constant, and keep the vertical part of the boundary at fixed temperature, say \( T(x = 0, t > 0) = 0 \).

If the kernel obeys \( \frac{\partial}{\partial x} g(x, X; t) = \lambda \frac{\partial^2}{\partial x^2} g(x, X; t) \) for all \( t > 0 \) and is constructed to satisfy the boundary conditions at \( x = 0 \), namely

\[
g(x = 0, X; t) = 0 \quad \text{for all} \quad t > 0 \quad \text{(and for all} \ X)\]

and also has the initial behavior

\[
\lim_{t \to 0} g(x, X; t) = \delta(x - X) \quad \text{for all} \quad x, X > 0
\]

then the solution to the problem in question is again given by an integral over the boundary.

\[
T(x, t) = \int_0^\infty g(x, X; t) T(X, 0) \, dX
\]

Only that part of the boundary with nonvanishing values of \( T \) contributes to the boundary integral.

So, how do we find the kernel? The simplest technique to use here is “the method of images.” The kernel \( g_0(x, X; t) \) can be thought of as the temperature at location \( x \) at any later time \( t > 0 \) that results from a \( t = 0 \) Dirac delta “pulse” located initially at \( X \), since for that initial temperature distribution

\[
T(x, t) = \int_\infty^{-\infty} \frac{e^{-(x-x)^2/(4\lambda t)}}{\sqrt{4\pi \lambda t}} \delta(x-X) \, dx = \frac{e^{-(x-X)^2/(4\lambda t)}}{\sqrt{4\pi \lambda t}}
\]

It is often possible to construct a kernel in the presence of fixed boundaries through the clever arrangement of such initial pulses, of various signs and strengths, subject to the restriction that any extra pulses must not be “inside” the region of interest. A few examples should aptly make this last point.

62
Consider further the L-shaped boundary \( \{ x = 0, t > 0 \} \cup \{ x \geq 0, t = 0 \} \), with \( T (x > 0, t = 0) \) arbitrary and with \( T (x = 0, t > 0) = 0 \), or \( g (x = 0, X; t > 0) = 0 \), as expressed above. The latter boundary condition on \( g (x, X; t) \) is satisfied by placing a positive pulse at \( X \) and a negative pulse at \(-X\). The negative pulse may be thought of as an inverted “image” of the positive pulse with respect to the vertical part of the boundary (but not the horizontal)! Thus

\[
g_D (x, X; t) = \frac{e^{-(x-X)^2/(4\lambda t)}}{\sqrt{4\pi \lambda t}} - \frac{e^{-(x+X)^2/(4\lambda t)}}{\sqrt{4\pi \lambda t}} \quad \text{for all } \quad t > 0. \tag{49}
\]

The subscript “\( D \)” indicates this kernel satisfies (homogeneous) Dirichlet boundary conditions at \( x = 0: g_D (x = 0, X; t > 0) = 0 \). Clearly the PDE is solved by this combination since it is satisfied by each of the terms: \( \frac{\partial}{\partial t} \frac{e^{-(x-X)^2/(4\lambda t)}}{\sqrt{4\pi \lambda t}} = \lambda \frac{\partial^2}{\partial x^2} \frac{e^{-(x-X)^2/(4\lambda t)}}{\sqrt{4\pi \lambda t}} \). We then consider \( T (x, t > 0) \) as given by

\[
T (x, t) = \int_0^\infty g_D (x, X; t) T (X, 0) \, dX
= \frac{1}{\sqrt{4\pi \lambda t}} \int_0^\infty \left( e^{-(x-X)^2/(4\lambda t)} - e^{-(x+X)^2/(4\lambda t)} \right) T (X, 0) \, dX
\]

What about the initial conditions for \( x > 0 \)? These are OK because

\[
\lim_{t \to 0} g_D (x, X; t) = \delta (x - X) - \delta (x + X) = \delta (x - X) \quad \text{for all } \quad x, X > 0
\]

The second Dirac delta does not contribute to integrals over the positive half-line. That is to say, the initial image pulse is not within the region of interest. Thus

\[
\lim_{t \to 0} \int_0^\infty g_D (x, X; t) T (X, 0) \, dX = \int_0^\infty \delta (x - X) T (X, 0) \, dX = T (x, 0)
\]

as required.

**HW Problem 9:** Assume Dirichlet boundary conditions on the L-shaped boundary \( \{ x = 0, t > 0 \} \cup \{ x \geq 0, t = 0 \} \), and solve \( \frac{\partial}{\partial t} T (x, t) = \lambda \frac{\partial^2}{\partial x^2} T (x, t) \) with the following boundary values. Keep the vertical part of the boundary at a fixed constant temperature, say \( T = 0 \), and consider the various cases:

\[
\begin{align*}
T (x > 0, t = 0) &= T_0 \quad \text{a constant} \\
T (x > 0, t = 0) &= T_0 e^{-x/L} \\
T (x > 0, t = 0) &= T_0 e^{-(x-a)^2/L^2}
\end{align*}
\]

**Solution:** TO BE WRITTEN ...

As another possibility involving this same L-shaped boundary, we consider Neumann boundary conditions on \( \{ x = 0, t > 0 \} \) and Dirichlet on the remainder of the boundary, \( \{ x \geq 0, t = 0 \} \). For simplicity, take the Neumann conditions to be homogeneous: \( \frac{\partial}{\partial x} T (x, t > 0) \big|_{x = 0} = 0 \). It is easy to see that by simply changing the sign of the image in the previous kernel, we obtain the appropriate Green function.

\[
g_N (x, X; t) = \frac{e^{-(x-X)^2/(4\lambda t)}}{\sqrt{4\pi \lambda t}} + \frac{e^{-(x+X)^2/(4\lambda t)}}{\sqrt{4\pi \lambda t}} \quad \text{for all } \quad t > 0. \tag{50}
\]

with \( T (x, t) \) for later times given in terms of the initial values for \( x > 0 \) by

\[
T (x, t) = \int_0^\infty g_N (x, X; t) T (X, 0) \, dX
= \frac{1}{\sqrt{4\pi \lambda t}} \int_0^\infty \left( e^{-(x-X)^2/(4\lambda t)} + e^{-(x+X)^2/(4\lambda t)} \right) T (X, 0) \, dX
\]
To verify that this is correct, we perform the following checks.

\[
\lim_{t \to 0} g_N(x, X; t) = \delta(x - X) + \delta(x + X) = \delta(x - X) \quad \text{for all} \quad x, X > 0
\]

thereby guaranteeing that \(\lim_{t \to 0} \int_0^\infty g_N(x, X; t) T(X, 0) \, dX = T(x, 0)\). Also

\[
\lim_{x \to 0^+} \frac{\partial}{\partial x} T(x, t) \quad = \quad \lim_{x \to 0^+} \int_0^\infty \frac{\partial}{\partial x} g_N(x, X; t) T(X, 0) \, dX
\]

\[
= \lim_{x \to 0^+} \frac{1}{\sqrt{4\pi t \lambda}} \int_0^\infty \frac{\partial}{\partial x} \left( e^{-\frac{(x-X)^2}{4t\lambda}} + e^{-\frac{(x+X)^2}{4t\lambda}} \right) T(X, 0) \, dX
\]

\[
= \lim_{x \to 0^+} \frac{1}{\sqrt{4\pi t \lambda}} \int_0^\infty \left( e^{-\frac{(x-X)^2}{4t\lambda}} + e^{-\frac{(x+X)^2}{4t\lambda}} \right) T(X, 0) \bigg|_{X=\infty}^{X=0} \frac{\partial}{\partial x} T(X, 0) \, dX
\]

where we have integrated by parts in the last step. Now, provided \(T(X, 0)\) is well-behaved enough to discard the end-point contributions resulting from the integration by parts, i.e. provided we impose the rather tepid condition \(\lim_{x \to 0^+} \left( e^{-\frac{(x-X)^2}{4t\lambda}} + e^{-\frac{(x+X)^2}{4t\lambda}} \right) T(X, 0) \bigg|_{X=\infty}^{X=0} = 0\), we have

\[
\lim_{x \to 0^+} \frac{\partial}{\partial x} T(x, t) \quad = \quad \lim_{x \to 0^+} \frac{1}{\sqrt{4\pi t \lambda}} \int_0^\infty \left( e^{-\frac{(x-X)^2}{4t\lambda}} + e^{-\frac{(x+X)^2}{4t\lambda}} \right) \frac{\partial}{\partial x} T(X, 0) \, dX
\]

\[
= 0
\]

Thus the Neumann conditions on the vertical boundary are satisfied. In practice, it is usually a trivial matter to check whether the discarded terms vanish, as assumed.

**HW Problem 10:** Assume Neumann boundary conditions on the vertical part of the L-shaped boundary \(\{x \geq 0, t = 0\} \cup \{x = 0, t > 0\}\), specifically \(\partial_x T(x, t > 0) \big|_{x=0} = 0\), and solve \(\frac{\partial}{\partial x} T(x, t) = \lambda \frac{\partial^2}{\partial x^2} T(x, t)\) with the following Dirichlet conditions on the horizontal part.

\[
T(x > 0, t = 0) = T_0 \quad \text{a constant}
\]

\[
T(x > 0, t = 0) = T_0 e^{-x/L}
\]

\[
T(x > 0, t = 0) = T_0 e^{-x^2/L^2}
\]

**Solution:** TO BE WRITTEN ...

Discuss the Dirichlet problem for the “slab” \(0 \leq x \leq L\) with boundary \(\{x = 0, t > 0\} \cup \{0 \leq x \leq L, t = 0\} \cup \{x = L, t > 0\}\). Obtain

\[
g_{D,N}(x, X; t) = \frac{1}{2L} \left\{ \vartheta \left( \sqrt{\frac{x^2 - X^2}{L^2}} \right) - \frac{\pi \lambda t}{L} \vartheta \left( \sqrt{\frac{x^2 + X^2}{L^2}} \right) \right\}
\]

**HW Problem 11:** For the slab problem, with \(T(x, t > 0) \big|_{x=L} = 0, T(0 < x < L, t = 0) = T_0\), show that

\[
T(x, t) = \int_0^L g_D(x, X; t) T(X, 0) \, dX
\]

where

\[
g_D(x, X; t) = \frac{1}{2L} \left\{ \vartheta \left( \sqrt{\frac{x^2 - X^2}{L^2}} \right) - \frac{\pi \lambda t}{L} \vartheta \left( \sqrt{\frac{x^2 + X^2}{L^2}} \right) \right\}
\]

gives the result in Nearing, §10.2, Eqn(14).

**Solution:** TO BE WRITTEN ...
HW Problem 12: Under what conditions is it true that the total “heat energy” within an interval $x_{\text{left}} < x < x_{\text{right}}$ is conserved, i.e. that

$$\frac{d}{dt} \int_{x_{\text{left}}}^{x_{\text{right}}} T(x, t) \, dx = 0$$

Discuss higher dimensional problems with rectangular symmetry.

Helmholtz equation in $n$ spatial dimensions

This results from the heat equation by supposing separation of the $t$ variable from the spatial variables. Say: $T(r, t) = T_k (r) e^{-k^2 \lambda t}$

$$(\nabla^2 + k^2) T_k (r) = 0$$

(51)

In terms of the spatial variables, this is an elliptic PDE whose solutions are uniquely determined by specifying either Dirichlet or Neumann (or a mixture, but not both; discuss “Robin” BCs) conditions on a closed spatial boundary.

Another way to view the Helmholtz equation is to think of it as an eigenvalue problem for the Laplacian. For the given boundary conditions, the allowed values of $-k^2$ constitute the eigenvalues, with $T_k (r)$ the eigenfunctions.

$$\nabla^2 T_k (r) = -k^2 T_k (r)$$

(52)

Now, in $n$ spatial dimensions with spherical boundary conditions, it is convenient to separate the Laplacian into radial and angular parts. The radial part is

$$\nabla^2 f(r) = \left( \frac{\partial^2}{\partial r^2} + \frac{n - 1}{r} \frac{\partial}{\partial r} \right) f(r)$$

The angular part is what remains.

$$\left( \nabla^2 - \frac{\partial^2}{\partial r^2} - \frac{n - 1}{r} \frac{\partial}{\partial r} \right) T(r) \equiv \frac{1}{r^2} (-L^2) T(r)$$

The “angular momentum” invariant $L^2$ in $n$ dimensions has periodic eigenfunctions that depend on $n - 1$ angular variables. These may be defined (e.g. as in A. Sommerfeld): $\theta, \phi_1, \cdots, \phi_p$, with $p = n - 2$.

(Conveniently, $p = 1$ in 3 dimensions.)

$$
\begin{align*}
x_1 &= r \cos \theta \\
x_2 &= r \sin \theta \cos \phi_1 \\
x_3 &= r \sin \theta \sin \phi_1 \cos \phi_2 \\
&\vdots \\
x_{p+1} &= r \sin \theta \sin \phi_1 \cdots \sin \phi_{p-1} \cos \phi_p \\
x_{p+2} &= r \sin \theta \sin \phi_1 \cdots \sin \phi_{p-1} \sin \phi_p
\end{align*}
$$

where $0 \leq r < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi_j \leq \pi$, for $j < p$, and finally $-\pi \leq \phi_p \leq \pi$.

In any case, the eigenvalues of $L^2$ on the $n - 1$ sphere swept out by the various angles are encoded in

$$
\begin{align*}
L^2 Y_\ell (\theta, \phi) &= \ell (\ell + n - 2) Y_\ell (\theta, \phi) \quad \text{for integer } \ell = 0, 1, 2, \ldots \\
&= \ell (\ell + p) Y_\ell (\theta, \phi)
\end{align*}
$$

where we have suppressed labels that distinguish the various independent functions for a given $\ell$ (i.e. there is a degeneracy). For example, in 3 dimensions, $Y_\ell (\theta, \phi) \rightarrow Y_{\ell m} (\theta, \phi)$, $-\ell \leq m \leq \ell$ in integer steps. The “quantization” of the eigenvalues arises from requiring periodic and finite solutions in all of the angular variables.

65
In particular, there is always one solution for any integer \( \ell \) which is independent of the \( \phi \) (cf. \( m = 0 \) in 3 dimensions) and which obeys the equation
\[
-L^2 Y_\ell (\theta) = \frac{1}{\sin^2 \theta} \frac{d}{d\theta} \left( \sin^2 \theta \frac{d}{d\theta} Y_\ell (\theta) \right) = -\ell (\ell + p) Y_\ell (\theta)
\]
It is quite easy to find a generating function for these. (We suppress an implicit \( p \) dependence in the functions on the RHS!)
\[
\frac{1}{(1 + s^2 - 2s \cos \theta)^{p/2}} = \sum_{\ell=0}^{\infty} s^\ell Y_\ell (\theta)
\]
sol clearly the functions are polynomials in \( \cos \theta \). They are known in the literature as “Gegenbauer” polynomials (Wien. Akad. 70 (1875)): \( Y_\ell (\theta) = C_\ell^{p/2} (\cos \theta) \). Note the normalization we have chosen. At \( \theta = 0, \cos \theta = 1 \),
\[
\frac{1}{(1 + s^2 - 2s)^{p/2}} = \sum_{\ell=0}^{\infty} s^\ell Y_\ell (\theta = 0)
\]
so that
\[
Y_\ell (\theta = 0) = p (p + 1) \cdots (p + \ell - 1) / \ell!
\]
Evaluating a few gives
\[
Y_0 = 1, \quad Y_1 = p \cos \theta, \quad Y_2 = \frac{1}{2} p ((p + 2) \cos^2 \theta - 1), \quad Y_3 = \frac{1}{6} p (p + 2) ((p + 4) \cos^2 \theta - 3) \cos \theta
\]
\[
Y_4 = \frac{1}{24} p (p + 2) ((p + 4) (p + 6) \cos^4 \theta - 6 (p + 4) \cos^2 \theta + 3), \quad \text{etc.}
\]
and we check that these are indeed eigenfunctions of the differential equation given above with the appropriate eigenvalues. In three dimensions, where \( p = 1 \), these are just the Legendre polynomials, \( P_\ell (\theta) = Y_\ell (\theta) \mid_{p=1} \). In this case the normalization at \( \theta = 0 \) is particularly simple: \( P_\ell (\theta = 0) = 1 \).

Various of the above statements can also be checked in the familiar framework of rectangular Cartesian coordinates.
\[
r = \sqrt{\sum_{j=1}^{n} x_j^2}, \quad L_{jk} = -i \left( x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} \right) = -L_{kj}
\]
\[
\nabla^2 = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}, \quad L^2 = \frac{1}{2} \sum_{j,k=1}^{n} L_{jk} L_{jk}
\]
We work out another form for \( L^2 \)
\[
L^2 = -i \sum_{j,k=1}^{n} \left( x_j \frac{\partial}{\partial x_k} \right) L_{jk} = - \sum_{j,k=1}^{n} \left( x_j \frac{\partial}{\partial x_k} \right) \left( x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} \right)
\]
\[
= - \sum_{j,k=1}^{n} \left( x_j \delta_{jk} \frac{\partial}{\partial x_k} + x_j^2 \frac{\partial^2}{\partial x_k^2} - x_j \delta_{kk} \frac{\partial}{\partial x_j} - x_j x_k \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \right)
\]
\[
^5 \text{Checking } \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin^2 \theta \frac{d}{d\theta} Y_\ell (\theta) \right) = -\ell (\ell + p) Y_\ell (\theta) \text{ for } \ell = 1, 2, 3, 4.
\]
\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin^2 \theta \frac{d}{d\theta} (\cos \theta) \right) = -(p + 1) (\cos \theta)
\]
\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin^2 \theta \frac{d}{d\theta} ((p + 2) \cos^2 \theta - 1) \right) = -2 (p + 2) ((p + 2) \cos^2 \theta - 1)
\]
\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin^2 \theta \left( (p + 4) \cos^2 \theta - 3 \right) \right) = -3 (p + 3) (\cos \theta ((p + 4) \cos^2 \theta - 3))
\]
\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin^2 \theta \left( ((p + 4) (p + 6) \cos^4 \theta - 6 (p + 4) \cos^2 \theta + 3) \right) \right) = -4 (p + 4) ((p + 4) (p + 6) \cos^4 \theta - 6 (p + 4) \cos^2 \theta + 3))
\]
66
That is to say
\[
L^2 = -r^2 \nabla^2 + (n - 1) \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + \sum_{j,k=1}^{n} x_j x_k \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k}
\]
\[
= -r^2 \nabla^2 + (n - 2) \mathbf{r} \cdot \nabla + (\mathbf{r} \cdot \nabla)^2
\]

Now, it may not be so obvious that this is the same as
\[
\left( \nabla^2 - \frac{\partial^2}{\partial r^2} - \frac{n - 1}{r} \frac{\partial}{\partial r} \right) T(\mathbf{r}) \equiv \frac{1}{r^2} \left( -L^2 \right) T(\mathbf{r}) \equiv \frac{1}{r^2} \nabla_0^2 T(\mathbf{r})
\]
rewritten as
\[
L^2 = -r^2 \nabla^2 + (n - 1) r \frac{\partial}{\partial r} + r^2 \frac{\partial^2}{\partial r^2}
\]
\[
= -r^2 \nabla^2 + (n - 2) r \frac{\partial}{\partial r} + \left( r \frac{\partial}{\partial r} \right)^2
\]

But it is! To see this, check it on an exponential.\(^6\)

\[
\exp(\mathbf{a} \cdot \mathbf{r}) = \exp\left( \sum_{i=1}^{n} a_i x_i \right)
\]

This exponential is nothing but a generating function for all monomials made out of rectangular Cartesian components. So, if it works here, it works in all cases where \(T(\mathbf{r})\) can be written as a sum (i.e. a multivariable power series) of such monomials. Thus we check
\[
\left( (n - 1) r \frac{\partial}{\partial r} + r^2 \frac{\partial^2}{\partial r^2} \right) e^{\mathbf{a} \cdot \mathbf{r}}
\]
against
\[
\left( (n - 1) \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + \sum_{j,k=1}^{n} x_j x_k \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \right) e^{\mathbf{a} \cdot \mathbf{r}}
\]

But clearly the two expressions are the same, for the two separate terms in each. That is
\[
\frac{\partial}{\partial r} e^{\mathbf{a} \cdot \mathbf{r}} = (\mathbf{a} \cdot \mathbf{r}) e^{\mathbf{a} \cdot \mathbf{r}} = \left( \sum_{j=1}^{n} a_j x_j \right) e^{\mathbf{a} \cdot \mathbf{r}} = \left( \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \right) e^{\sum_{i=1}^{n} a_i x_i}
\]
\[
\frac{\partial^2}{\partial r^2} e^{\mathbf{a} \cdot \mathbf{r}} = \left( \sum_{j=1}^{n} a_j x_j \right) \left( \sum_{k=1}^{n} a_k x_k \right) e^{\mathbf{a} \cdot \mathbf{r}} = \left( \sum_{j,k=1}^{n} x_j x_k \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \right) e^{\sum_{i=1}^{n} a_i x_i}
\]

For the case that the radial and angular variables also separate,

\[
T_k(\mathbf{r}) = f_\ell(kr) Y_\ell(\theta, \phi)
\]

Helmholtz’s equation reduces to
\[
\left( \nabla^2 + k^2 \right) T_k(\mathbf{r}) = Y_\ell(\theta, \phi) \left( \frac{\partial^2}{\partial r^2} + \frac{n - 1}{r} \frac{\partial}{\partial r} - \frac{\ell(\ell + n - 2)}{r^2} + k^2 \right) f_\ell(kr) = 0
\]

\(^6\)You can save a bit of writing here, and in the preceding formulas, if you adopt the convention that any repeated index is summed over all components, unless specifically forbidden to do so. Thus, for example, we have \(a_i x_i \equiv \sum_{i=1}^{n} a_i x_i\). This is Einstein’s summation convention. However, I will refrain from using the convention for the present discussion.
Note that if \( k \neq 0 \) this equation only depends on the combination \( kr \), a fact we have already taken advantage of in writing \( f(kr) \). (For the time being, I leave it to the student to consider \( k = 0 \).) If we define \( h \) by

\[
 f_\ell (r) = \frac{1}{r^{p/2}} h_\ell (kr)
\]

Helmholtz’s equation becomes Bessel’s equation for \( h \), with index \( \ell + p/2 \). This is the only way in which the number of spatial dimensions \( (n = 2 + p) \) enters into the radial problem. That is to say

\[
 \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + k^2 - \frac{(\ell + p/2)^2}{r^2} \right) h_\ell (kr) = 0
\]

Thus

\[
 f_\ell (r) = \frac{1}{r^{p/2}} \begin{cases} 
 J_{\ell+p/2} (kr) & \text{where } \ell = 0, 1, \ldots \\
 N_{\ell+p/2} (kr)
\end{cases}
\]

Observe the only difference between these radial functions in \( n \) and \( n + 2 \) spatial dimensions is the overall power of \( r \) and a shift in the Bessel function index. That is to say

\[
 f_\ell (r)|_{n+2} = \frac{1}{r} f_{\ell+1} (r)|_n
\]

In this sense, if you understand one case of even dimension, you understand them all! Ditto odd dimensions. But of course, the angular functions become more complicated as the number of dimensions increases.

**HW Problem 13:** Consider a two-dimensional circular disk of radius \( R \) with initial temperature

\[
 T(r < R, 0 \leq \theta \leq 2\pi, t = 0) = 0
\]

and subsequent boundary condition

\[
 T(r = R, 0 \leq \theta \leq 2\pi, t > 0) = T_0
\]

Show that the solution of the angle independent heat equation

\[
 \frac{\partial}{\partial t} T(r, t) = \lambda \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) T(r, t)
\]

which satisfies the aforementioned initial and boundary conditions is

\[
 T(r, \theta, t > 0) = T_0 \left( 1 - 2 \sum_{n=1}^{\infty} \frac{J_0 (\alpha_n r/R)}{\alpha_n J_1 (\alpha_n)} \times e^{-\alpha_n^2 \lambda t/R^2} \right)
\]

where \( \alpha_n \) is the \( n \)th positive zero of the cylindrical Bessel function, \( J_0 (\alpha_n) = 0 \).

**Solution:** TO BE WRITTEN ...

For reference purposes, we record here the expansion of a plane wave, moving “upward” along the polar axis, in terms of the \( Y_\ell (\theta) \)’s.

\[
 e^{ikr \cos \theta} = \frac{1}{2} \Gamma \left( \frac{p}{2} \right) \left( \frac{2}{kr} \right)^{p/2} \sum_{\ell=0}^{\infty} (2\ell + p) e^{i\ell \pi/2} Y_\ell (\theta) J_{\ell+p/2} (kr)
\]

Also recall \( \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \).
Of particular interest is the case of three spatial dimensions, for which case \( p = 1 \). In this case define the spherical Bessel functions as

\[
\begin{align*}
\tilde{j}_\ell (s) &= \sqrt{\frac{\pi}{2s}} j_{\ell + 1/2} (s) , \quad n_\ell (s) = \sqrt{\frac{\pi}{2s}} N_{\ell + 1/2} (s) \\
j_\ell (s) &= s^{\ell} \left( -\frac{1}{s} \frac{d}{ds} \right)^\ell \sin s \frac{1}{s} \cos \left( s - \frac{\pi}{2} (\ell - 1) \right) \\
&= \frac{s^\ell}{(2\ell + 1)!!} \left( 1 - \frac{s^2}{2(2\ell + 3)} + \cdots \right) \\
n_\ell (s) &= -s^\ell \left( -\frac{1}{s} \frac{d}{ds} \right)^\ell \cos s \frac{1}{s} \sin \left( s - \frac{\pi}{2} (\ell - 1) \right) \\
&= -\frac{(2\ell - 1)!!}{s^{\ell + 1}} \left( 1 + \frac{s^2}{2(2\ell - 1)} + \cdots \right)
\end{align*}
\]

The aforementioned plane wave expansion in this case becomes

\[e^{i k r \cos \theta} = \sum_{\ell = 0}^{\infty} (2\ell + 1) e^{i \ell \pi / 2} P_\ell (\cos \theta) j_\ell (k r)\]

**HW Problem 14:** Consider the solutions of the radial Helmholtz equation in three dimensions, written as

\[H_1 \psi_1 (s) = \psi_1 (s) , \quad H_1 = D_1^+ D_1^- , \quad D_1^\pm = \frac{l}{s} \pm \frac{d}{ds}\]

with \( \psi_1 (s) \) either of

\[u_1 (s) = sj_1 (s) \quad \text{or} \quad w_1 (s) = sh_1 (s) = s (j_1 (s) + in_1 (s))\]

Use the relations

\[\psi_{\ell + 1} = D_1^+ \psi_\ell , \quad \psi_{\ell - 1} = D_1^- \psi_\ell\]

as well as

\[u_0 (s) = \sin s , \quad w_0 (s) = -ie^{is}\]

to determine both the small \( s \) and the large \( s \) behaviors of the \( \psi_\ell \). That is, show

\[u_1 (s) \underset{s \to 0}{\sim} \frac{j_1 (s)}{(2l + 1)!!} , \quad w_1 (s) \underset{s \to 0}{\sim} -i(2l - 1)! \frac{e^{is}}{s^l}\]

\[u_1 (s) \underset{s \to \infty}{\sim} \sin \left( s - \frac{l\pi}{2} \right) , \quad u_1 (s) \underset{s \to \infty}{\sim} -i \exp \left( s - \frac{l\pi}{2} \right)\]

where the double factorial is defined as \((2l + 1)!! = 1 \times 3 \times 5 \times \cdots \times (2l - 1) \times (2l + 1)\) and by convention \((2l - 1)!!|_{l=0} = 1\). *Solution:* TO BE WRITTEN ...

**HW Problem 15:** Consider the radial Helmholtz equation in \( n \) spatial dimensions.

\[
\left( \frac{d^2}{ds^2} + \frac{n - 1}{s} \frac{d}{ds} - \frac{l(l + n - 2)}{s^2} + 1 \right) f_1 (s) = 0
\]

Rescale the radial function \( f_1 (s) = s^a u_1 (s) \). Determine the exponent \( a \) so as to eliminate the first derivative term in the Helmholtz equation. Then factor the resulting equation as we did for \( n = 3 \), writing it in the form

\[u_1 (s) = H_1 u_1 (s) = D_1^+ D_1^- u_1\]

69
Obtain explicit expressions for \( D_L^k \), and check that they agree with those in HW#15 for \( n = 3 \).

**Solution:** TO BE WRITTEN ...

Another especially interesting case arises in the study of conformal transformations, which ultimately leads to \( p = 2 \). In this regard, we first note that Chebyshev polynomials \( T_n(z) \) are generated by

\[
\frac{1 - h z}{1 - 2 h z + h^2} = \sum_{n=0}^{\infty} h^n T_n(z)
\]

Explicitly\(^7\)

\[
T_n(z) = \frac{1}{2} \delta_{n,0} + \frac{n}{2} \sum_{k=1}^{\frac{n}{2}} \frac{(-1)^k (n - k - 1)! (2z)^{n-2k}}{k! (n - 2k)!} + 2^{n-1} z^n
\]

But the generating function we are interested in, when \( p = 2 \) in the above, is

\[
\frac{1}{1 - 2 h z + h^2} = \frac{1}{1 - h z} \sum_{n=0}^{\infty} h^n T_n(z) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} h^{n+k} z^k T_n(z)
\]

\[
= \sum_{m=0}^{\infty} h^m \sum_{j=0}^{m} z^j \Gamma (m-j) T_m-j(z)
\]

\[
= \sum_{m=0}^{\infty} h^m C_m^1(z)
\]

More appropriately, the last sum involves Gegenbauer polynomials of a particular kind, namely \( C_m^1(z) \) where in general the \( C_m^\lambda(z) \) are generated by

\[
\frac{1}{(1 - 2 h z + h^2)^\lambda} = \sum_{m=0}^{\infty} h^m C_m^\lambda(z)
\]

Explicitly, we have\(^8\)

\[
C_m^\lambda(z) = \sum_{k=0}^{\frac{m}{2}} \frac{\Gamma (\lambda + m - k)}{\Gamma (\lambda)} \frac{(-1)^k (2z)^{m-2k}}{k! (m - 2k)!} \Gamma (a_n) = a (a+1) \cdots (a+n-1) = \Gamma (a+n)/\Gamma (a), (1)_n = n!
\]

So there are simple linear relations between the Gegenbauer and Chebyshev polynomials. In particular, as we have exhibited above,

\[
C_m^1(z) = \sum_{j=0}^{m} z^j T_{m-j}(z)
\]

\(^7\)The floor function \([x]\), also called the greatest integer function, gives the largest integer less than or equal to \( x \). In many computer languages, the floor function is called the integer part function and is denoted \( \text{int}(x) \). The name and symbol for the floor function were coined by K. E. Iverson.

\(^8\)Normally here I would use the Pochhammer symbol as in Mathews and Walker. \( (a)_n = a (a+1) \cdots (a+n-1) = \Gamma (a+n)/\Gamma (a) \) (1)_n = n!. But this notation is confusingly the same as that used for the “falling factorial” \( (x)_n = x (x-1) \cdots (x-n+1) \) falling factorial and should probably be avoided, were it not for the fact that it is used in Abramowitz and Stegun, as well as M&W. Most other sources denote the Pochhammer symbol as the “rising factorial” \( x^{(n)} = x (x+1) \cdots (x+n-1) \) rising factorial a.k.a. Pochhammer symbol.