HW Problem 18:

A generating function for one-dimensional simple harmonic oscillator energy eigenfunctions may be written as

\[ e^{-\frac{1}{2}s^2} G(x, s) = \exp \left( -\frac{1}{2}x^2 + 2sx - s^2 \right) = \sum_{n=0}^{\infty} \frac{s^n H_n(x) e^{-\frac{1}{2}x^2}}{n!} \]

in terms of the Hermite polynomials \( H_n(x) \). From this expression, show the orthogonality and normalization of the eigenfunctions is

\[ \int_{-\infty}^{+\infty} H_n(x) H_k(x) e^{-x^2} \, dx = 2^n n! \sqrt{\pi} \delta_{nk} \]

Do this by first evaluating

\[ \int_{-\infty}^{+\infty} G(x, s_1) G(x, s_2) e^{-x^2} \, dx \]

and then series expanding the result in powers of both \( s_1 \) and \( s_2 \).

HW Problem 19:

Rescale the variables in the previous problem so that the \( n \)th eigenfunction satisfies the equation

\[ \left( -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + \frac{1}{2} M \omega^2 x^2 \right) \psi_n(x) = \hbar \omega \left( n + \frac{1}{2} \right) \psi_n(x) \]

thereby obtaining an explicit generating function for the \( \psi_n(x) \). Give explicit expressions for the \( \psi_n(x) \) in terms of \( H_n \). What is the normalization \( \int_{-\infty}^{+\infty} |\psi_n(x)|^2 \, dx \) of your eigenfunctions?